Theory of a Free-Electron Laser with a Gaussian Optical Undulator

JUAN C. GALLARDO, RICHARD C. FERNOW, ROBERT PALMER, AND C. PELLEGRINI

Abstract—We present theoretical studies of the possibility of a free-electron laser in the XUV and soft X-ray region of the spectrum, using a counterpropagating CO₂ laser beam as an undulator. A one-dimensional model is used to evaluate the single-pass power gain of such a device. The detrimental effects due to the energy spread, emittance of the electron beam, and the diffraction of the electromagnetic undulator are explicitly incorporated in the formalism. An oscillator experiment has also been considered. We derive the solutions of the optical pulse evolution equation in the weak field, low gain, and long electron pulse regime.

As an example, we apply our results to state-of-the-art electron beams and conclude that they would not satisfy the requirements to operate a free-electron laser in a self-amplified spontaneous emission regime. On the other hand, preliminary results for the oscillator experiment suggest that moderate improvements in the quality of the electron beam and the CO₂ laser energy would make it feasible to obtain sufficient small-signal gain to overcome the expected mirror losses in the 10-50 nm range of wavelengths.

I. INTRODUCTION

In a conventional [1] free-electron laser (FEL), coherent emission of radiation is due to the bunching of a relativistic electron (e⁻) beam propagating along a static periodic magnetic field, the undulator. The Lorentz force on the electrons makes them follow transverse oscillatory orbits and thus to radiate (spontaneous emission). The combined effect of this radiation and the undulator produces a periodic ponderomotive potential in the longitudinal direction which forces the electrons to bunch on the wavelength scale and to radiate coherently.

An e⁻ beam of energy \( E = \gamma mc^2 \) traveling through a circularly polarized undulator of period \( \lambda_0 \) produces, from noise, a coherent laser field of wavelength (resonance condition)

\[
\lambda = \frac{\lambda_0}{2\gamma} \left(1 + K^2\right)
\]

where \( K \), the undulator parameter, is

\[
K = \frac{e\lambda_0 B}{2\pi mc^2}
\]

and \( B \) is the undulator peak magnetic field; typically, \( K \approx 1 \) and \( \lambda_0 \approx 2 \text{ cm} \).

Scaling down the wavelength range of operation of an FEL is a challenging task that has been theoretically explored by several research groups [2] and has been the object of several review conferences [3]. There are many factors that contribute to make a short-wavelength FEL (XUV and soft X-ray) experiment difficult to realize. The single-pass gain is proportional to the wavelength \( \lambda \) of the produced radiation, the number of periods in the undulator, and the number of electrons in the pulse, and it is also inversely proportional to the normalized emittance of the electron beam. Thus, the decrease of the gain as we reduce the wavelength can be compensated by increasing the number of periods or the total number of particles in the pulse or by decreasing the emittance of the beam. These conflicting requirements have motivated a number of projects where specially designed photocathode injectors are incorporated into conventional accelerators to provide the high brightness and quality (low energy spread and emittance) needed to operate an FEL in the high-gain regime. This is required due to the poor mirror quality at these wavelengths. Some of these projects would use a storage ring [2], [3], while others like the projects at Los Alamos [4] and at Brookhaven [5] are designed to make use of an RF Linac. In these proposed experiments, long undulators and high-energy electrons are required. We will not discuss here the concurrent stringent requirements on undulator design [6] and mirror quality at these high frequencies [7].

Instead, we turn to a different approach to obtain coherent XUV laser light, namely, to reduce the wiggle period by means of a real electromagnetic wave. The idea of using an electromagnetic wave as an undulator has been entertained by several research groups [8], and some of these experiments are either in the early stage of their actual construction or in the final stage of their design studies. This paper describes the theoretical effort carried out at Brookhaven [5] to investigate the feasibility of building an XUV and soft X-ray FEL using the high-brightness and high-quality electron beam as well as a powerful CO₂ laser to be provided by the Accelerator Test Facility (ATF), presently under construction. The expected properties of the electron beam are summarized in Table I.

It is apparent from (1) that to obtain coherent radiation in the region of 500 Å or shorter, we must either increase the energy \( \gamma \approx 3 \times 10^5 \) (\( \lambda_0 \approx 1 \text{ cm} \)) or reduce the undulator period \( \lambda_0 \approx 10 \mu \text{m} \) (\( \gamma \approx 10 \)). Thus, the idea

Manuscript received March 9, 1988. This work was supported in part by the U.S. Department of Energy under Contract DE-AC02-76-CH00016.

The authors are with Brookhaven National Laboratory, Upton, NY 11973.

IEEE Log Number 8821738.

0018-9197/88/0800-1557$01.00 © 1988 IEEE
of using a counterpropagating CO₂ laser beam has the advantage of relaxing the need for high-energy electrons, but at the same time it introduces a number of additional difficulties which mainly result in further constraints on the quality of the electron beam [9] and the stability of the phase and amplitude of the CO₂ laser pulse (undulator).

A counterpropagating CO₂ laser beam is seen by the relativistic electrons as a circularly polarized undulator of half the period of the actual wavelength of the pump laser. Hence, the resonance condition differs from (1) by a factor of $\frac{1}{2}$, namely,

$$\lambda = \frac{1}{4} \frac{\lambda_0}{\gamma^2} \left( 1 + K^2 \right).$$  

There are several experiments underway which use an electromagnetic wave as an undulator [8]. The idea is an old one, going back to the original work by Maday [1]. Later on, the two-stage FEL concept was developed where the laser field of an FEL oscillator (first stage) is made to interact with the same electron beam that generates it, thereby producing radiation at wavelength $\lambda = 1/4\lambda_0^2 \left( \lambda_0/2\gamma^2 \right)$.

Our single-pass gain calculations parallel the one presented in several papers by researchers at the Max-Planck Institute für Quantenoptik and Center of Advanced Studies of the University of New Mexico [10]. In this work, we have added the inhomogeneous broadening effects introduced by nonideal electron beams.

The pump laser field is assumed to be a standard focused TEM₀₀ mode [11]. Therefore, the intensity of the magnetic field on axis changes as we move along the $z$ axis, and consequently, so does the undulator parameter $K$. The magnetic field experienced by the electrons increases as they approach the beam waist and then decreases again as they move away from it according to $B_0/(1 - i(z/z_0))$. $z_0$ is the Rayleigh length of the Gaussian wave and the spot size is $\omega_0 = z_0/\lambda_0$. An additional effect to be considered is the dependence on the transverse dimensions of the laser beam, namely, $B(r, z) = B_0 \exp \left( -\left( r^2/2\omega(z)^2 \right) \right)$ with $\omega(z) = \omega_0 \sqrt{1 + (z/z_0)^2}$.

For the moment, we are only concerned with the longitudinal effects which modulate the transverse velocity of the electrons in amplitude and phase, and we leave the treatment of a full three-dimensional model to a forthcoming paper.

The energy in the laser beam is defined as $W = P T_L = (c/8) B_0^2 \omega_0^2 \tau_L$, where $\tau_L$ is the temporal length of the laser pulse. Expressing the amplitude of the field in terms of the energy of the laser, we can write the undulator parameter as

$$K^2(z) = 0.69 \frac{\lambda_0^2}{\omega(z)^2} \frac{W[\text{Joule}]}{l_L[\text{cm}]} \frac{1}{\gamma^2}.$$  

As we will see in the next section, the $z$ dependence of $K^2(z)$ can be accounted for explicitly in the laser field evolution equation; therefore, in what follows, we will use the value of the undulator parameter at the beam waist $K_0^2 = (0.69) 2\pi (W/N_L) q_L$ where we have introduced $q_L = N_{\lambda_0}/2 z_0$; this parameter measures the length of the interaction region in units of the Rayleigh length of the pump laser [12].

The parameter that determines the gain of the device is given by

$$j = 8\pi^2 \frac{\varepsilon_e^2 N \lambda_0^2}{mc^2} \frac{1}{\gamma^3} K_0^2 N \rho$$  

where $\rho$ is the electron particle density. For a cylindrically symmetric beam of radius $\sigma_r(z)$ and angular divergence $\theta_0$, we have

$$\sigma_r^2(z) = \sigma_r^2(0) + \theta_0^2, \quad \theta_0^2 = \varepsilon_e \beta^* \gamma, \quad \sigma_r^2(0) = \varepsilon_e \beta^*.$$  

Averaging over the interaction region, we get $\langle \sigma_r^2 \rangle = (4/3)(\varepsilon_e \gamma)^2 \beta^*$. The electron particle density can be rewritten in terms of the total number of electrons $\mathcal{N}$ in each pulse of temporal length $\tau_r$ as $\rho = \mathcal{N}/A_0 c \tau_r = \mathcal{N} \gamma/[1/4(3\pi\epsilon_0 c^3 \beta^* \tau_r)]$. Replacing both $\rho$ and $K_0^2$ in the definition of $j$, we obtain

$$j = 4.6 \times 10^{-11} \frac{\eta_r q_L N \lambda_0[\text{cm}]}{l_L[\text{cm}]} \frac{W[\text{Joule}]}{l_L[\text{cm}]} \frac{\mathcal{N}}{\varepsilon_e[\text{cm} \cdot \text{rad}]} \frac{1}{\gamma^2}$$  

with $l_L = c \tau_r$ and $\eta_r = N \lambda_0/2 \beta^*$. In this 1-D model, we are leaving out a filling factor $F = A_r/A_t$ where $A_r$ ($A_t$) is the transverse area of the electron (XUV laser) beam. The most efficient coupling happens when $F = 1$ or, in other words, when the spot size of the laser beam is of the same order as the electron beam radius [13]. Assuming a Gaussian beam for the laser light, the diffraction-limited angle is $\theta_0 = w(z)/z = \lambda/\pi w_0$, where $w_0 = (1/2\pi) \sqrt{L_c \lambda}$ is the XUV laser beam spot size and $L_c$ is the cavity length. A filling factor $F = 1$ ($\sigma_r = w_0$) yields $(4/3)(\varepsilon_e \gamma)^2 \beta^* = (1/4\pi^2) L_c \lambda$. Solving for the length of the cavity $L_c$, we obtain the optimum value $L_c = 16\pi^2 \varepsilon_e \gamma \beta^*/3\gamma \lambda$, which maximizes the electron–XUV laser beams interaction. There are two addi-
tional conditions on $N$, the number of periods of the undulator: 1) the energy spread of the $e$ beam $\Delta \gamma / \gamma < 1 / 2N$, and 2) short pulse effects are minimized if $\mu _{e} = N \lambda _{e} / l < 1$. For the set of parameters we are interested in, $N$ turns out to be $\approx 100$. It can be seen from (5) that the gain will increase with $N$ (as long as it satisfies the above constraints), which in turn decreases the value of $K_0$ and consequently makes less important the higher harmonic contribution.

II. BASIC THEORY OF A LONG-PULSE FREE-ELECTRON LASER

In this section, we will briefly describe the multimode theory of an FEL with an electron pulse much larger than the slippage distance $l$, $>> s$ or long-pulse regime. Since our discussion is one dimensional, we only include the longitudinal structure of the electron and laser beams. The effects of the finite transverse dimensions of the input CO$_2$ laser (undulator) will be negligible as long as the electron beam radius is smaller than its spot size ($\sigma _{e} << \omega _{0}$). The diffraction effects of the stimulated radiation are also neglected; however, some phase shift effects ($i (m + 1/2)$ arc-cos $\Phi _{e} (\tau )$) can be easily incorporated in the formalism.

Our approach follows closely the Colson formulation [1], [14]–[15]. Some modifications of the canonical equations of an FEL are required in the case of an electromagnetic wave undulator. First, we present the single mode theory; next, the equations for the multimode case will be derived in Section II-B; and in the last section, we will show the evolution equation of the laser field in a resonator configuration.

A. Single-Mode Theory

A particularly useful form of describing the electron dynamics in the combined undulator and laser field is to introduce the electron phases $\xi (\tau ) = k_2 - \omega _{0} t + k_{0} z + \omega _{0} t$ where $\tau = ct / L$ is a normalized time over $L = N \lambda _{0}$, the length of the undulator, and $(k, \omega _{0}, k_{0}, \omega _{0})$ are the wave vector and angular frequency of the radiation field and the counterpropagating CO$_2$ laser wave, respectively. The phase velocity is $v(\tau ) \equiv d\xi / dt = L(k + k_{0}) \beta _{2} - L(k - k_{0})$, the normalized electron velocity is $\beta _{e}$, and $\gamma = 1/\sqrt{1 - (K(\tau )^2 / \gamma _{R}^2) - \beta _{2}^2}$ is obtained under the assumption that the transverse trajectory of the electrons is solely determined by the undulator magnetic field. The resonance condition is found by imposing $\nu = 0$, that is, $\beta _{2} (\tau ) = (k - k_{0}) / (k + k_{0})$, which is (3) after solving for $\lambda = 2 \pi / k$. Using $d\beta _{2} / dt = 1 / \beta _{2} (1/\gamma _{R} (d\gamma _{R} / dt) - \nu)$, in combination with $(d/dt) v(\tau ) = \nu = L(k + k_{0}) (d/dt) \beta _{2}$, the Lorentz equation

$$
\frac{d}{d\tau} \gamma = \frac{e}{mc_{e} \gamma} K_{0} \text{Re} \left\{ \frac{e^{i\nu(\tau)} E(\tau)}{1 + iq_{L}(2\tau - 1)} \right\}
$$

we can write

$$
\dot{\nu}(\tau) = \frac{L(k + k_{0})}{\beta _{2} \gamma c} \left( 1 + K_{0}^{2} \right) \frac{e}{mc_{e} \gamma} \left\{ K_{0} \text{Re} \left\{ \frac{e^{i\nu(\tau)} E(\tau)}{1 + iq_{L}(2\tau - 1)} \right\} \right.
$$

$$
\left. - \frac{L(k + k_{0})}{2\beta _{2} \gamma c} \frac{d}{d\tau} |K(\tau)|^2 \right\}.
$$

In the above equations, we have used $q_{L} = N \lambda _{0} / 2 \gamma _{R}$, introduced in the previous section, $\text{Re} \{ \cdot \}$ denotes the real part, and $K(\tau )^2 = K_{0}^2 (1 + iq_{L}(2\tau - 1))$ is the undulator parameter including the diffraction of the input laser. Let us define the dimensionless optical field

$$
a(\tau) = \frac{8\pi e K_{0} \gamma _{R}^{2} E(\tau)}{mc_{e} \gamma_{R} K_{0}^{2}} \frac{1}{1 - \frac{(1 + K_{0}^2)}{\gamma _{R}^2}},
$$

the electron energy at resonance $\gamma _{R}$, and $\delta = 4\pi N K_{0}^{2} / \beta _{2} (1 + K_{0}^2)$. $\delta$ represents the “taper” of the undulator and its origin can be traced back to the diffraction of the CO$_2$ laser beam. Notice that $\delta$ is proportional to the energy of the laser pulse [see (4)]. The electron phase is readily found by integrating (9) twice:

$$
\gamma(\tau) = \gamma_{0} + \nu_{0} \tau - \Delta_{1} (\eta(\tau) + \text{arctan} q_{L})
$$

$$
+ \int_{0}^{\tau} dr' (\tau - r') \text{Re} \left\{ \frac{a(r') e^{-i\nu(r')}}{1 + iq_{L}(2\tau - 1)} \right\}
$$

where $\Delta_{1} = \delta / 2 \nu_{0}$, $\eta(\tau) = \text{arctan} q_{L}(2\tau - 1)$, and $\gamma_{0}$, $\nu_{0}$ are the electron initial conditions. The optical field is governed by the wave equation, which after using the slow wave approximation and a coordinate change $\tau \rightarrow \tau = ct / L$ and $z \rightarrow Z = z - ct$, reads

$$
\frac{d}{d\tau} a(\tau) = -j \left\{ \frac{e^{-i\nu(\tau)}}{1 - iq_{L}(2\tau - 1)} \right\}_{\tau_{0}, \nu_{0}}
$$

where $j$ is the current parameter defined in (7) and $\langle \cdot \cdot \cdot \rangle$ represents an average over the initial phases $\gamma_{0}$ and velocities $\nu_{0}$ of the electrons in a section of the beam of length $\lambda$.

The coupled system of (11) and (12) describes simply and accurately the physics of the FEL interaction. We have left out in these equations a factor $(1 - (\nu(\tau) / 2 \pi N))$ which is close to unity unless the electrons lose an appreciable amount of energy during the interaction.

In weak field $(|a(1)| << \pi)$, we can linearize (11) and (12) obtaining an integral equation for the laser field:

$$
\frac{d}{d\tau} a(\tau) = \frac{ij}{2} \left\{ \int_{0}^{\tau} dr' (\tau - r') a(r')
$$

$$
\cdot \frac{e^{-i\nu_{0}(\tau - r')}}{1 - iq_{L}(2\tau - 1)} \frac{e^{i\nu_{0}(\tau - r')}}{1 + iq_{L}(2\tau - 1)} \right\}_{0, \nu_{0}}
$$

(13)
where the average over the electron phase \( \phi_0 \) has been explicitly [14], [15] done and only the dependence on the initial electron velocity \( v_0 \) remains to be determined.

We denote with \( g(v_0) \) a velocity distribution function with \( \langle \cdots \rangle = \int_{-\infty}^{\infty} dv_0 g(v_0) \langle \cdots \rangle \). If we consider a Gaussian distribution in energy, then

\[
ge_{\theta}(v) = \frac{e^{-((v-v_0)^2/2\sigma_e^2)}}{\sqrt{2\pi}\sigma_e} \quad \text{if} \quad \nu \leq \nu_0,
\]

\[
ge_{\theta}(v) = \frac{\nu_0 - \nu}{\sigma_e} \quad \text{if} \quad \nu > \nu_0.
\]

Notice that this distribution is asymmetric because for any angle \( \Delta \theta \), the electrons can only be slowed down and \( \sigma_t = \frac{1}{2} \Delta \theta^2/2\sigma_e^2 \). Evaluating the average over \( v_0 \) in (13), we obtain

\[
a(1) = a(0) + \frac{i}{2} \int_{(2\pi - 1)} e^{i\Delta \theta(\tau)} (1 - i \xi_0(2\tau - 1))
\]

\[
\times \int d\tau' e^{-i\nu\tau'} a(\tau') e^{i\nu\tau'}
\]

\[
e^{-\Delta \theta(\tau)} \frac{e^{-\Delta \theta(\tau)}}{1 + i \xi_0(2\tau - 1)} (1 - i \xi_0(\tau - \tau'))
\]

\[
e^{-\Delta \theta(\tau)} \frac{e^{-\Delta \theta(\tau)}}{1 + i \xi_0(2\tau - 1)} (1 - i \xi_0(\tau - \tau'))
\]

\[
(1 - i \xi_0(2\tau - 1))(1 + i \xi_0(2\tau' - 1))
\]

\[
(1 - i \xi_0(\tau - \tau'))
\]

\[
(1 - i \xi_0(\tau - \tau'))
\]

represents the phase shift and amplitude modulation introduced by the CO2 beam diffraction. This is the mass-shift effect discussed in [10].

We have solved numerically the integral equation (16) and evaluated the power gain defined by \( G = (|a(1)|^2)/(|a(0)|^2) - 1 \) for a set of parameters that correspond to the electron beam to be provided by the ATF and an electromagnetic wave undulator. The expected parameters of both beams are summarized in Table II. In Fig. 1, we plot the gain surface as a function of the resonance parameter \( v_0 \) and the energy spread \( \sigma_e \) for a fixed current \( j \). The various plots correspond to different \( q_L \) (we can vary the focusing of the CO2 undulator) and \( \eta_e = 0.2 \) (fixed external electron beam focusing). We see from the expression for \( j \) in (7) that the gain increases with \( q_L \). However, the "tapering effect" is rather insensitive to small \( q_L \) and becomes negligibly small \( O((1/q_L^2)) \) for large \( q_L \), and hence there is an optimum value, which we found to be \( q_L \approx 2.6 \). Also, the position of the maximum of the gain curve moves to larger values of resonance parameter \( (\nu_{\text{max}} - \Delta \theta) \).

In Fig. 2, we plot the gain surface as a function of \( r_0 \) and angular spread \( \sigma_\theta \) for fixed current parameter \( j \) and \( \eta_e = 0.2 \); the various curves correspond to increasing values of \( q_L \). The appropriate value of \( \eta_e = 0.2 \) can be obtained from the condition that the electron beam radius must be much smaller than the CO2 spot size \( r_a \leq 0.1 \omega_0 \) or otherwise an additional reduction of the gain is introduced by the transverse decay of the undulator field. Using the definitions for \( q_L \) and \( \eta_e \), we find the relation

\[
q_L \approx 0.01 \frac{\gamma v_0}{\epsilon_0 \eta_e}.
\]

The maximum FEL gain predicted for the ATF accelerator is 80 percent. It is evident from the above results that further developments in the brightness of the electron beam and the energy of the CO2 laser are required to achieve a reasonable exponential gain. Nevertheless, the value of the gain in this analysis suggests that with a moderate improvement of the accelerator performance, we could build an oscillator experiment for these short wave-
lengths. Discussions of the pertinent theory is in Section III.

B. Multimode Theory

The formalism developed in the previous subsection has only included a single longitudinal mode of frequency \( \omega \), the carrier wave. However, we must extend the formalism to allow for multiple modes for at least two reasons. First, the understanding of the start-up of a short-pulse FEL and its ultimate coherent capabilities can only be achieved by a theory with a spectrum of longitudinal modes. Second, the electron beam produced by an RF accelerating cavity, such as a Linac, microtron, or storage ring, is made of a train of \( \approx 1 \) nm long micropulses. This induces a similar pulse structure in the produced light with a corresponding Fourier transform-limited spectrum. The laser wave shows a prominent mode, the carrier wave, surrounded by a spectrum of frequencies with a sufficiently long coherence length to maintain the validity of the slowly varying amplitude and phase approximation. To generalize the formalism to the multimode case, we follow the arguments in [1] and [14] and define the laser field \( a(z, \tau) \) and the electron phase \( \chi(z, \tau) \) with spatial dependence \( z \).

The general set of FEL equations reads

\[
\frac{\partial}{\partial \tau} a(z, \tau) = -j(z) \left( e^{-j(z, \tau)} \right)_{\partial \omega} \tag{18}
\]

and

\[
\left( \frac{\partial}{\partial \tau} - s \frac{\partial}{\partial z} \right) \chi(z, \tau) = \nu(z, \tau)
\]

\[
\left( \frac{\partial}{\partial \tau} - s \frac{\partial}{\partial z} \right) \nu(z, \tau) = \text{Re} \left\{ a(z, \tau) e^{j(z, \tau)} \right\} \tag{19}
\]
where the $z$ dependence of the current parameter $j(z)$ describes the macroscopic envelope of the electron beam and the slippage distance $s = (1 - \beta_e)L = N\lambda$ is a characteristic length in the longitudinal direction representing the distance the carrier wave phase front moves ahead of a sample of electrons initially coincident with the phase front after traversing the undulator. This slip of the laser pulse over the electron pulse is a purely kinematic effect due to the different longitudinal velocity between the electron beam ($\beta_e < 1$) and the wave front. The slippage leads to many interesting dynamical effects in the electron beam–radiation field interaction, which are correctly and accurately described by the system of differential equations (18) and (19).

A formal solution of the Lorentz equations for the electron phase variable is

$$
\dot{\zeta}(z, \tau) = \dot{\xi}_0 + r_0 \tau - \Delta_\eta(\eta(\tau) + \arctan \eta \xi)L \int_0^\tau \Re \left\{ \frac{a(z + s(\tau - \tau'), \tau')}{1 + iq(2r' - 1)} \right\} \cdot e^{i\frac{\xi(z + s(\tau - \tau'), \tau')}{\tau}}.
$$

(20)

The difference with (11) for the single-mode case is that the integrand above exhibits explicitly the dynamical effects caused by the slippage, i.e., bunching function at position $z$ and time $\tau$ is determined by the laser field and bunching at previous times $\tau' < \tau$ (causality) and at an advance position $z + s(\tau - \tau')$, which is the distance the electron will slip backward with respect to the laser wave in the time interval $(\tau - \tau')$.

In the weak field case $|a(z, \tau)| \ll 1$, we expand

$$
e^{-j(\zeta(z, \tau))} \approx e^{-j[\xi_0 - \Delta_\eta s(\tau) + i\Delta_\eta s(\tau')]} \times \left[ 1 - i \int_0^\tau d\tau' (\tau - \tau') \cdot \Re \left\{ \frac{e^{i\eta(\tau - \tau') - \Delta_\eta s(\eta(\tau'))}}{1 + iq(2r' - 1)} \right\} \cdot a(z + s(\tau - \tau'), \tau') \right].
$$

(21)

and keep the first-order terms in (18) and (20). Substituting (21) in (18), we obtain an integral equation for the laser field of the form

$$a(z, 1) = a(z, 0) + \frac{1}{2} ij(z) \left\{ \int_0^1 \int_0^\tau d\tau' (\tau - \tau') \cdot \frac{e^{-in(\tau - \tau') \xi(z + s(\tau - \tau'), \tau')}}{(1 - iq(2r' - 1))(1 + iq(2r' - 1))} \cdot a(z + s(\tau - \tau'), \tau') \right\}_{r_0}.
$$

(22)

This equation is the basic expression of our one-dimensional multimode model of an FEL oscillator to be discussed in the next section.

III. OSCILLATOR CONFIGURATION: LONG-PULSE PROPAGATION THEORY

The discussion up to this point has been restricted to an amplifier configuration with a continuous electron beam. As we have seen in the previous section, with the performance expected at the ATFV [16] and the CO_2 beam, it will not be possible to operate an SASE-FEL; hence, we turn to an oscillator configuration. To study this alternative, we have to address two related problems concerning the repetition rate of the CO_2 laser ($\sim 1 \text{ Hz}$) and the pulse structure of the electron beam. We want to arrange the geometry of the CO_2 optics to have the laser pulse of approximate length $\tau_1 \sim 6 \text{ ps}$ interact with all the micropulses of the electron beam train provided by the ATF-Linac.

In [10], it has been suggested to use a ring cavity for the CO_2 pulse with an amplifier to compensate for the losses at the mirrors and the pump depletion due to the FEL interaction [17]. Fig. 3 shows the schematics of a possible oscillator design for the CO_2 laser (undulator). The ATF-Linac will produce a macropulse lasting 5 $\mu$s composed of 5 ps pulses separated by 12.5 ns. The optical path in the ring resonator of the CO_2 pulse is chosen in such a way that each electron micropulse will overlap both the undulator field and the XUV laser pulse at the beginning of the undulator. This provides the feedback mechanism which will, if the system is above threshold, turn on the FEL.

To discuss an FEL oscillator, we need to consider two additional parameters $Q$ and $\delta L$. $Q$ is the losses' coefficient at the mirrors and the cavity; it is a phenomenological parameter that could be chosen to be complex to describe dielectric surface mirrors. $\delta L$ is the detuning cavity length; it represents the necessary shortening of the optical cavity, due to lethargy, with respect to the nominal synchronous length $L_c = \frac{1}{\pi}T$ where $T$ is the separation time between electron pulses. The expected range [18] of values for $\delta L$ over which the laser will operate in very small $0 \lesssim |\delta L| \lesssim 50 \mu$m. This phenomenon has become known as laser lethargy and it is a direct consequence of the pulse structure of the electron beam and the slippage between the light and the electrons. The laser pulse suffers a reshaping of its macroscopic envelope due to the nonlinear FEL gain mechanism by which the power extracted from the electrons is mainly used to amplify the rear end of the laser pulse with the consequent reshaping of it and a slight decrease of the group velocity of the XUV pulse. If the length of the cavity $L_c$ is set equal to half the distance between electron micropulses, the laser pulse will walk off the trailing edge of the electron pulse over many passes and it will eventually die due to the losses.

There have been several approaches [19] for
studying the FEL pulse propagation problem. We will follow the space–time approach which has been put forward by one of the authors [23]. Our aim is to develop a semi-analytical theory of a long-pulse FEL oscillator in the small-signal and low-gain regime. As we have stated in the preceding section, (22) is the fundamental FEL equation. Its numerical solution \( a(z, 1) \) is used as the initial condition for the following roundtrip in the cavity, after including the losses \( Q \) and taking into account the detuning of the cavity \( \delta L \). To write the differential equation for the laser field for a time scale corresponding to a roundtrip in the optical cavity, we define \( t' = n(2Lc/c) = nT \), and use the boundary conditions \( A(z + \delta L, t' + T) = e^{-i\lambda(z, \tau)} A(z, t') A(z, t') \) with \( a^{(0)}(z, \tau = 0) = A(z, t') \) and \( a^{(0)}(z, \tau = 1) = A(z + \delta L, t' + T) \). Expanding the left-hand side of (22) and keeping the first term in \( \delta L \) and \( T \), we obtain

\[
\frac{\delta L}{\delta z} \frac{\partial}{\partial z} + T \frac{\partial}{\partial t'} A(z, t') = -\frac{1}{2} QA(z, t')
\]

\[
+ \frac{1}{4} j(z) \left[ G_1(v_0, \Delta, q_0) + sG_2(v_0, \Delta, q_0) \frac{\partial}{\partial z} + \frac{1}{2} s^2G_3(v_0, \Delta, q_0) \frac{\partial^2}{\partial z^2} \right] A(z, t')
\]

(23)

where

\[
G_1(v_0, \Delta, q_0) = 2i \int_0^1 d\tau \int_0^1 d\tau' (\tau - \tau')^2 \frac{e^{-i\lambda(\tau - \tau')}}{(1 - i\delta L(2\tau - 1))} \frac{e^{-\Delta g(\tau - \tau')}}{(1 - i\delta L(2\tau' - 1))}
\]

(23a)

\[
G_2(v_0, \Delta, q_0) = i \frac{\partial}{\partial v_0} G_1(v_0, \Delta, q_0)
\]

and

\[
G_3(v_0, \Delta, q_0) = \frac{\delta^2}{\delta v_0^2} G_1(v_0, \Delta, q_0).
\]

(23b)

In the derivation of (23), we have assumed that the laser field does not change significantly in a single pass (low-gain regime), and consequently \( a(z, \tau) \) has been pulled out from the integrals that define the \( G \) functions. \( G_i \) is the complex gain function and the other two are higher order corrections due to the finite length of the electron pulse. One important assumption in the derivation is that the slippage distance is very small compared to all the characteristic lengths on a macroscopic scale of the FEL oscillator problem, namely, the length of the optical cavity \( L_{\text{cavity}} \), undulator \( L_{\text{undulator}} \), and the electron pulse \( L \). The pulse effects will be important when \( L_{\text{cavity}}, L_{\text{undulator}} \gg L \), and the long pulse refers to the situation when \( s << L \). Fig. 4 displays plots of the \( G \) functions for the parameters shown in Table II. For our particular example, the maximum of the gain function \( \text{Re} \{ G_1 \} \) occurs at \( v_0 \approx 25 \).

An important parameter to characterize the pulse effects is the coupling parameter \([24, 21, 23] \) or slippage parameter \([20] \) defined as \( \mu_c = s/L \). Most FEL experiments, and in particular, the one planned at BNL, are in the region \( \mu_c << 1 \); therefore, we are justified to approximate the current function by \( j(z) = j_0(1 - (1/2)(z^2/L^2)) \).

Changing to more appropriate variables \( z^* = z/L \), \( t^* = (1/4)(tj_0/T) \), \( \Theta = 4\delta L/sj_0 \), the above evolution equa-
which is a Schrödinger-like equation with a non-Hermitian Hamiltonian. The non-Hermiticity of the Hamiltonian operator implies that the eigenfunctions of the corresponding time-independent Schrödinger equation do not form a basis in the ordinary sense (except in the particular case $G_{\Delta}(v_0, \Delta, q_L) - \Theta = 0$), but in conjunction with the eigenfunctions of the hermitian Hamiltonian $H_0$, they form a biorthogonal set of eigenfunctions [25], i.e.,

$$\hat{H} A_n = \lambda_n A_n, \quad \hat{B}^* B_m = \lambda^*_m B_m$$

with $\int_\infty^\infty d\xi A_n^*(\xi) B_m(\xi) = \delta_{n,m}$. We refer the reader for the details of the calculation to the works in [23].

The eigenfunctions $A_n(\xi)$ are a collection of interacting "cold" longitudinal modes of the optical cavity with the property of being self-reproducing after each roundtrip in the cavity up to a complex number, the eigenvalue $\lambda_n$. The real part of $\lambda_n$ represents gain if it is positive or absorption if it is negative. These combinations of longitudinal modes, called "supermodes," were first introduced in [21] and later on were used to analyze the storage ring FEL experiment at Orsay [26]. This concept has also been used in three-dimensional studies of FEL's [22].

We summarize below the more important results of this theoretical study of an FEL in the small-signal and low-gain regime for a long electron beam pulse. 1) The eigenvalues of the stationary Schrödinger-like equation are

$$\lambda_n(v_0, \mu, Q, j_0, \Delta, q_L) = -n + \frac{1}{2} \omega_0 + \Gamma$$

with

$$\omega_0 = \mu \sqrt{G(v_0, \Delta, q_L) G_3(v_0, \Delta, q_L)}$$

and

$$\Gamma = G(v_0, \Delta, q_L) - \frac{2Q}{j_0} - \frac{1}{2} \frac{(G_3(v_0, \Delta, q_L) - \Theta)^2}{G_3(v_0, \Delta, q_L)}.$$

The gain growth rate of each supermode is given by $\text{Re} \lambda_n$, and clearly the fundamental mode has the highest gain.

$\text{Re} \lambda_0$ depends quadratically on $\delta L$ in agreement with the qualitative experimental behavior [22]; the region of cavity length variation with positive gain is found from the condition $\text{Re} \lambda_0 = 0$.

2) We turn now to the eigenfunctions $A_n(\xi)$. The harmonic oscillator nature of the Hamiltonian allows us to write a normalized $A_n(\xi)$ as

$$A_n(\xi) = \frac{1}{\sqrt{n!2^{n} \kappa \pi \sigma_L^2}} \exp - \frac{1}{2 \sigma_L^2} (\xi - \xi_0)^2$$

with $\sigma_L^2 = \mu \sqrt{G_3(v_0, \Delta, q_L) G_3(v_0, \Delta, q_L)}$, $\xi_0 = - (G(v_0, \Delta, q_L) - \Theta) / G_3(v_0, \Delta, q_L)$.

The fundamental state, the supermode with maximum gain, is

$$A_0(\xi) = \frac{1}{\sqrt{\sqrt{\pi} \sigma_L}} e^{-\frac{1}{2} (\xi - \xi_0)^2}$$

which is a Gaussian function with dispersion $\sigma_L^2$ and with an amplitude which is a decreasing exponential of $(G(v_0, \Delta, q_L) - \Theta)$. Therefore, the laser field will be larger when $\xi_0 = - (G(v_0, \Delta, q_L) - \Theta) / G_3(v_0, \Delta, q_L)$ $G_3(v_0, \Delta, q_L)) = 0$, i.e., $\text{Re} G_3(v_0, \Delta, q_L) = \Theta$ and $\text{Im} G_3(v_0, \Delta, q_L) = 0$, which is the synchronism condition to achieve optimal overlap of the electron and laser pulses (see [23] for a discussion of this point). The above condition reflects the lethargic behavior or slowing down of the laser pulse and the corresponding shortening of the optical cavity. When this condition is satisfied, we not only ensure the timing of arrival of both the laser pulses bouncing off the mirrors of the cavity and the electron pulse coming from the RF Linac, but also that both pulses stay overlapped during the whole trip through the undulator.

In real space, the laser field is a Gaussian function with dispersion

$$\sigma_L^2 = \sigma^2 L = s L \frac{\sqrt{G_3 G_3^*}}{G_1}.$$

The spectrum can be calculated taking the Fourier transform of $A_n(\xi)$; the spectrum profile is also a Gaussian function with dispersion in wavelength $\sigma^2$ dual to $\sigma_L^2$, i.e.,

$$\sigma^2 = \frac{\lambda^2}{2 \pi \sigma_L^2} = \frac{\lambda^2}{2 \pi \sqrt{s L}}.$$

The typical width of the spontaneous emission curve is $\Delta \lambda / \lambda = 1 / N$; hence, the spectrum width of the fundamental supermode (which is the state the system evolves to and has the smallest width) is $(1 / 2 \pi) \sqrt{(s / L)}$, narrower than the spontaneous incoherent spectrum.

IV. CONCLUSIONS

Our analysis indicates that an FEL operating in the SASE regime is not feasible with the expected brightness and quality at the ATF-Linac. The oscillator experiment looks more promising, and only moderate improvements on the quality of the electron beam will make it theoretically possible to increase the gain to be above threshold. Further studies are needed concerning the resonator con-
figuration, assessment of mirror losses at this short wavelength, as well as its lifetime due to radiation damage before a more definite conclusion can be reached.

All the attempts to produce FEL coherent radiation at short wavelengths suffer the drawback of rather intense background spontaneous emission in the harmonics of the resonance wavelength, i.e., $\lambda_n = \lambda/n$ (n = 1, 3, 5, \ldots) which tends to damage the multilayer dielectric mirrors tuned to the fundamental frequency. The intensity emitted into a frequency interval $d\omega$ and a solid angle $d\Omega$ is calculated in a rather straightforward way by means of the Lienard–Wiechert fields and is proportional to [27]

$$\frac{d^2I}{d\Omega d\omega} \propto \frac{2(c\pi)^2}{c^2 \gamma^2} \frac{K^2}{(1 + K^2)} \sum_{n=1}^{\infty} \left( \frac{n}{n_{\text{ce}}} \right)^2$$

with $n_{\text{ce}} = \left[ n - (a/2\gamma^2\omega_0)/(1 + K^2) \right] N_\pi$. Clearly, at an equal number of periods and undulator parameter, an FEL with a permanent magnet undulator will produce a background radiation $= (30)^2$ stronger than an FEL using an optical undulator. In conclusion, we wish to stress that the last mentioned point of mirror radiation damage makes the idea of an FEL with a CO$_2$ undulator and low-energy electron beam a very promising alternative to the more standard long permanent magnet and high-energy electron beam as generators of coherent radiation in the XUV and soft X-ray region of the spectrum.

ACKNOWLEDGMENT

The authors would like to thank the members of the Center for Accelerator Physics (CAP) at BNL for many critical comments and contributions to this study. In particular, J. C. Gallardo is especially indebted to H. Kirk for his help and valuable information on the ATF project.

REFERENCES


Juan C. Gallardo attended the Universidad Nacional de Córdoba, Argentina from 1960 to 1964 and received the "Licenciado en Física" degree in physics in December 1964, and the Ph.D. degree in physics in June 1970 from the Belfer Graduate School of Science, Yeshiva University, New York, NY. His dissertation work was on the theory of strong interactions.

From 1980 to 1987 he was a Research Physicist at the Quantum Institute, University of California, Santa Barbara, working on free-electron laser theory. Last year he joined the Center for Accelerator Physics, Brookhaven National Laboratory, Upton, NY. He is engaged in research on theoretical problems in electron–laser interaction for the production of coherent radiation.

Dr. Gallardo is a member of the American Physical Society.

Richard C. Fernow was born in Newark, OH, on February 5, 1947. He received the B.S. degree in physics from Ohio University in 1969. He did his graduate dissertation on experimental particle physics and received the Ph.D. degree in 1973.

He worked as a Postdoctoral Fellow at the University of Michigan from 1973 to 1978. His primary contributions were in the development of polarized target systems. He joined the Physics Department at Brookhaven National Laboratory, Upton, NY, in 1978 where he is currently employed as a staff Physicist. His early work at Brookhaven was on particle physics experiments, calculations of magnetic fields, and superconducting magnet development. His current research interests are in the study of new methods for accelerating electrons using lasers, and in the generation of coherent radiation using electron-laser interactions.

Dr. Fernow is a member of the American Physical Society.

Robert Palmer immigrated from England in 1960 to work at the Brookhaven National Laboratory (BNL), Upton, NY. He worked on many high-energy experiments including the "Ω" discovery, the neutral current discovery, and the observation of the first charmed baryon. In 1980 while working on the mothballed ISABELLE accelerator, he designed a superconducting magnet, many of whose features have become incorporated into the proposed Superconducting Super Collider. He was appointed to the High Energy Physics Advisory Panel in 1981, and two years later was made Associate Director for High Energy Research at Brookhaven National Laboratory. For the past few years he has commuted between BNL and California’s Stanford Linear Accelerator Center (SLAC) where he is working on ideas for future electron–positron linear colliders.

C. Pellegrini, photograph and biography not available at the time of publication.