Strong sextupole focusing in planar undulators

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(Received 28 October 1993)

The range of parameters used in both working and proposed free electron lasers (FELs) has greatly expanded in recent years. Future machines are envisioned as fourth generation light sources providing researchers with high brightness, short wavelength, coherent radiation [see Ref. [1]: Report on the Workshop on Fourth Generation Light Source SLAC, Stanford CA, USA (February, 1992)]. Short wavelength operation, along with the long undulator required to produce high brightness photon beams in these devices, places tight requirements on the electron beam quality. In particular, the need to maintain a well focused beam is critical to successful operation of such an FEL. This paper examines the use of alternating gradient (AG) sextupole focusing in planar undulators. The equations of motion for an electron in an undulator field with a strong sextupole component are examined. Analytic as well as smooth approximation solutions are provided for AG sextupole focusing. It is shown that the mean electron longitudinal velocity can be kept constant through each focusing and defocusing section, but that the velocity differs between these sections. The effects of this stepwise velocity modulation as well as the beam size variation are explored computationally and compared to theory. Examples using the proposed SLAC 4 nm FEL, the UCLA 10.6 μm FEL as well as a Paladin based device are also given.

1. Introduction

Planar undulators, especially hybrid designs [2], offer a number of advantages over other schemes: high peak fields, simplicity, beamline accessibility and flexibility [3]. However, planar, as opposed to helical, undulators lack symmetric transverse focusing: planar undulators only have “natural” focusing in the transverse dimension normal to the undulation plane. This natural focusing, however, can be shared with the undulation dimension by appropriate magnetic field shaping to introduce a sextupole component. Because the average (over an undulator period) longitudinal velocity of the electron is constant over a betatron oscillation, natural focusing preserves the phase relation between the optical field and the electron wiggle motion.

Recent designs call for free electron lasers using undulators tens of meters long [4]. These designs also call for electron beam sizes which cannot be achieved without a strong focusing mechanism. Other undulators operating at low beam energies (in the collective or Raman regime) require confinement of the beam over short distances due to space charge induced divergence. Divergence of the electron beam in an FEL, regardless of its source, reduces efficiency through a lowering in beam density, and by consequent reduction in the beam overlap with the radiation field.

Relevant length scales for focusing include the undulator period, radiation wavelength and electron beam parameters. These effects can be examined using the dimensionless FEL parameter, ρ [5]. In SI units we may write [6]

$$\rho \approx 0.14 \frac{J^{1/3} B_u^{2/3} \lambda_u^{4/3}}{\gamma_t},$$

where $J$ is the beam current density and the remainder of the notation is as given in Table 1 [7]. It should be noted that this expression is strictly valid for undulators with only natural focusing. Other authors have derived expressions for $\rho$ which include focusing in an attempt to find optimal operating conditions [8]. Here we assume a given FEL design, and concern ourselves with the details of the effects of a particular type of focusing (strong sextupole) on its performance.

In the high gain regime of an FEL, the power gain e-folding length is approximately

$$L_\ast = \frac{\lambda_u}{\sqrt{3} 4\pi \rho}.$$  

The output power, $P$, as a function of the distance along the undulator, $z$, may be expressed as $P = P_0 \exp(z/L_\ast)$ where $P_0$ is the initial power (at $z = 0$). Thus a decrease in the beam current density, $J$, will
Table 1
Notation used in this paper

<table>
<thead>
<tr>
<th>Notation</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Undulator magnetic field</td>
<td>$B_u$</td>
</tr>
<tr>
<td>Normalized undulator field</td>
<td>$b_{ij} = (e/mc^2)B_u$</td>
</tr>
<tr>
<td>Electron charge</td>
<td>$e$</td>
</tr>
<tr>
<td>Electron mass</td>
<td>$m$</td>
</tr>
<tr>
<td>Speed of light</td>
<td>$c$</td>
</tr>
<tr>
<td>Undulator period</td>
<td>$\lambda_u$</td>
</tr>
<tr>
<td>Undulator wavenumber</td>
<td>$k_u = 2\pi/\lambda_u$</td>
</tr>
<tr>
<td>Electron beam energy [$me^2$]</td>
<td>$\gamma$</td>
</tr>
<tr>
<td>Radiation wavelength</td>
<td>$\lambda$</td>
</tr>
<tr>
<td>Radiation wavenumber</td>
<td>$k_r = 2\pi/\lambda_r$</td>
</tr>
<tr>
<td>Beam emittance (normalized, rms)</td>
<td>$\epsilon_n$</td>
</tr>
</tbody>
</table>

adversely affect gain. Maintaining and increasing the beam current density through focusing is thus highly desirable.

A general feature of high gain free electron lasers is that if variations occur on a scales shorter than a gain length, performance is adversely affected. This rule of thumb, along with a related requirement on the phase space of the electron beam, imposes a limit on the strength of focusing: $\beta > L_g$ where $\beta$ is the average inverse betatron wavenumber. The necessity for the electron beam to overlap well with the output radiation [9] can be expressed, for a beam with no external focusing, as a limit on the emittance:

$$\epsilon_n \leq \frac{\lambda_u}{4\pi \gamma}.$$  \hspace{1cm} (3)

When focusing is taken into account, a limit on the emittance as a function of the focusing strength can be given:

$$\epsilon_n \leq \frac{\beta \rho (1 + a_0^2/2)}{2\gamma}.$$  \hspace{1cm} (4)

This result is derived in the one dimensional limit assuming no energy spread, and can be useful for setting an upper limit on how strong a given FEL’s focusing channel can be. As the examples indicate, this expression is typically not the most stringent limit.

In subsequent sections of this paper we review FEL focusing, and derive the equations of transverse motion for strong sextupole focusing. We present a matrix formalism for describing this motion, and examine the effects of this focusing on the longitudinal dynamics of the electron beam. In order to quantitatively assess these effects, we present some examples using existing and planned FELs and examine the performance of these devices under strong sextupole focusing using computer simulations.

2. Focusing schemes

Various FEL focusing schemes have been considered and presented in the literature: quadrupole focusing [10], solenoidal confinement [11] and ion focusing [12,13]. Little experimental work has been performed on these various schemes with the exception of quadrupole focusing. Quadrupole focusing can be implemented in a number of ways. Typically, external quadrupole magnets in a FODO lattice are superimposed over the undulator field. External magnets require that no permeable materials be used in the undulator; hybrid undulators are not compatible with external magnets. Alternatively, the quadrupoles can be interspersed with undulator sections. Undulators with canted pole faces can also produce quadrupole fields. Regardless of the arrangement, quadrupole focusing differs from natural focusing in that it causes electron velocity modulation during betatron oscillations [14].

Sextupole focusing satisfies the phase preservation requirement of an FEL; however, if it is constant gradient, it is weak focusing. The idea of shaping the poles of a planar undulator to provide weak focusing has been examined in detail by Scharlemann [15]. Sextupole focusing avoids the problems and complexities associated with other focusing schemes, and can produce focusing up to the strength of the natural vertical focusing. This concept has proven itself in application [16]. The major drawback of this scheme is the weakness of focusing, as is discussed further below.

3. Strong sextupole focusing

Here we present a calculation of the equations of motion for alternating gradient focusing. The inspiration for this treatment comes from Scharlemann’s work on weak sextupole focusing and the calculation by Dattoli and Renieri using a smallness parameter to describe the effective horizontal defocusing sextupole strength [17].

Implementing strong focusing with sextupoles, which requires alternating from focusing to defocusing sections, can overcome the natural focusing strength limitation. A set of poles which focus in, say, $x$ and defocus in $y$ is followed by a set which defocuses in $x$ and focuses in $y$. This is analogous to strong focusing with quadrupoles. Since sextupole fields are quadratic, the off-axis orbit taken by the design electron moves through a region with a linear gradient, giving rise to a quadrupole-like focusing (this is an example of the feeddown effect in beam dynamics). One constraint is that the design orbit (and velocity) of the electrons must match closely in the two types of undulator sections. It has already been verified that this is the case in a weak focus (Scharlemann) and a weak defocus (Dattoli and Renieri). It is still necessary to show that the results are in fact valid for strong focusing [18].
The geometry used in the calculations is as follows: The beam propagates along the positive z-axis with the undulations occurring in the x-z plane and the undulator field along with natural focusing occurring in the y-z plane.

We begin, as Scharlemann did, with a normalized undulator magnetic field which satisfies the Maxwell equations and has a symmetric dependence on the transverse distance from the undulator axis:

\[
b = \frac{b_0 k_x}{k_y} \sinh(k_x x) \sin(k_y y) \cos(k_u z) \\
+ \frac{\gamma b_0}{2} \frac{k_x}{k_y} \sinh(k_x x) \cos(k_y y) \cos(k_u z) \\
- \frac{\gamma b_0 k_u}{k_y} \sinh(k_x x) \sin(k_y y) \sin(k_u z),
\]

where the notation is as given in Table 1. The Maxwell equations additionally require that

\[
k_u^2 = k_x^2 + k_y^2.
\]

When the focusing strength in one plane exceeds the natural focusing strength, the strong focusing regime is entered, and \(k(x, y) > k_u\), \(k(y, x)\) becomes imaginary.

A derivation of the field to arbitrary order is also possible, but analysis shows that the correction terms beyond second order are negligible (for known FEL parameters) [19]. For small \(k_x x\) and \(k_y y\) the field may therefore be approximated by

\[
b_x = b_0 k_x^2 x y \cos(k_u z), \\
b_y = b_0 \left(1 + \frac{k_x^2 x^2}{2} + \frac{k_y^2 y^2}{2}\right) \cos(k_u z).
\]

The electron equations of motion are straightforward to derive:

\[
\ddot{x} = \frac{c}{\gamma} \left(\dot{y} b_y - \dot{b} \dot{x}\right), \\
\ddot{y} = \frac{c}{\gamma} \left(\dot{x} b_x - \dot{b} \dot{y}\right), \\
\ddot{z} = \frac{c}{\gamma} \left(\dot{b} \dot{x} - \dot{x} \dot{b}\right).
\]

Here the dot is used to indicate a derivative with respect to time. A natural scale length of the problem is the undulator period; by separating the fast oscillations (those occurring at the undulator frequency) from the slow secular behavior, it is possible to simplify the equations. Following Scharlemann we define \(r(x, y, z) = r_0 + r_1\) where \(r_0\) is constant over the undulator period (the slow betatron oscillation) and \(r_1\) varies within a period (the fast undulator oscillation). Then, the equations of motion can be written, to leading order, as

\[
\ddot{x} = \frac{c}{\gamma} \frac{\dot{z}}{b_0}, \\
\ddot{y} = \frac{c}{\gamma} \frac{\dot{z}}{b_0} \\
\ddot{z} = \frac{c}{\gamma} \frac{\dot{b}_0}{b_0} \left(1 + \frac{k_x^2 x^2}{2} + \frac{k_y^2 y^2}{2}\right) \sin(k_u z).
\]

Averaging and simplification gives the desired solution for the equations of motion for the three cases of weak focusing, strong focusing in \(x\), and strong focusing in \(y\):

\[
\ddot{x}_0 + c^2 k_x^2 x_0 = 0 \\
\ddot{y}_0 + c^2 k_y^2 y_0 = 0 \\
\ddot{z}_0 + c^2 k_u^2 z_0 = 0 \\
\ddot{x}_0 - c^2 k_x^2 x_0 = 0 \\
\ddot{y}_0 + c^2 k_y^2 y_0 = 0
\]

Here the notation

\[
k_{\mu}(x, y) = \frac{b_0}{\sqrt{2} \gamma k_u} |k_{(x,y)}|
\]

has been introduced. The constraints on the transverse focusing betatron wavenumbers for these three cases follow from Eq. (6):

\[
k_{\mu}^2 + k_{\mu}^2 = \frac{e}{2m c^2 \gamma^2 b_0}, \\
k_{\mu}^2 - k_{\mu}^2 = \frac{e}{2m c^2 \gamma^2 b_0}, \\
-k_{\mu}^2 + k_{\mu}^2 = \frac{e}{2m c^2 \gamma^2 b_0}.
\]

Note that for the alternating gradient cases the focusing strengths relative to the natural case are \(k_x/\gamma k_u\) and \(k_y/\gamma k_u\) for the x and y directions, respectively. The above sets of equations can each be integrated by using the relation between the derivatives with respect to time and distance (z). Scharlemann has
shown that the additional term coming from the longitudinal acceleration (velocity modulation) does not contribute to the average focusing, and the relation \( d/dt = v \cdot \frac{d}{dz} \) is a good approximation. Now it remains to evaluate the average transverse velocity:

\[
\langle \beta^2_\perp \rangle = \frac{1}{c^2} \left( \langle \dot{x}^2 \rangle + \dot{y}^2 \right).
\]

(14)

The \( x_1 \) term is averaged by noting that \( \langle \sin^2(k_w x) \rangle = 1/2 \). Then, elimination of higher order terms leaves

\[
\langle \dot{x}^2 \rangle = \frac{c^2 b_0^2}{2 y^2 k_u^2} \left( 1 + k_x^2 x_\beta^2 + k_y^2 y_\beta^2 \right).
\]

(15)

This equation holds for all three cases considered. So, the average transverse velocities for each case become:

\[
\langle \beta^2_\perp \rangle = \frac{b_0^2}{2 y^2 k_u^2} \left( 1 + k_x^2 x_\beta^2 + k_y^2 y_\beta^2 \right) \quad \text{for} \quad k_{x,y} < k_u,
\]

(16a)

\[
\langle \beta^2_\perp \rangle = \frac{b_0^2}{2 y^2 k_u^2} \left( 1 + k_x^2 x_\beta^2 - k_y^2 y_\beta^2 \right) \quad \text{for} \quad k_{x,y} > k_u,
\]

(16b)

\[
\langle \beta^2_\perp \rangle = \frac{b_0^2}{2 y^2 k_u^2} \left( 1 - k_x^2 x_\beta^2 + k_y^2 y_\beta^2 \right) \quad \text{for} \quad k_x > k_u,
\]

(16c)

where \( x_\beta \) and \( y_\beta \) are the amplitudes of the transverse betatron oscillation, i.e. \( x_0 = x_\beta \sin[k_\beta z + \phi_x] \) for a focusing section, or \( x_0 = x_\beta \sinh[k_\beta z + \phi_x] \) for a defocusing section. It is now possible to see that \( \langle \beta^2_\perp \rangle \) is constant in each case. That is, an electron’s velocity averaged over an undulator period is constant through a betatron oscillation within a particular case of focusing. This indicates that the (longitudinal) phase of the electrons within a ponderomotive “bucket” is not modulated. Hence, one would expect that sextupole focusing is not deleterious to electron bunching and FEL gain. In fact, as was discussed in the Introduction, the gain is expected to be higher since the beam density remains greater under focus.

The above statements hold true for weak (constant gradient) focusing. For strong focusing they apply only within a particular focusing section. In the transition from, for example, horizontal focusing (Eq. (16b)) to defocusing (Eq. (16c)), the velocity is not in general constant. Since it can be shown that the betatron amplitudes \( x_\beta \) and \( y_\beta \) are constant for each electron and remain the same across a lens boundary, one can see that \( \langle \beta^2_\perp \rangle \) is in general different in defocusing and focusing sections. It is not feasible to make the velocities equal in the two types of sections for all electrons; any realistic beam will have a spread in the betatron amplitudes. However, this shortcoming does not in itself necessarily imply that strong sextupole focusing is problematic.

Indeed, phase space mixing and detrapping are possible at the boundaries between a focusing and defocusing sections. This situation inspires an analogy to tapered undulators. Theory indicates that the tapering should be performed gradually, but practical considerations can necessitate stepped tapering. Likewise, it is expected that if the focusing is not too strong the FEL gain will not be adversely affected. A calculation of the effective step difference in energy (due to a commensurate change in longitudinal velocity) across a focusing/defocusing section boundary and comparison of this quantity to the ponderomotive “bucket height” (the maximum energy deviation of the trapped electrons) could be performed for any case of interest, in order to determine whether or not detrapping would be a problem in this scheme.

The variation of the electron phase across the focus–defocus boundary can provide a useful gauge of the extent of detrapping. This variation is similar to an effective energy spread. The maximum phase change, \( \Delta \psi \), can be estimated as

\[
| \Delta \psi | = \frac{k_t}{2} \int \Delta \left( \beta^2_\perp \right) \, dz.
\]

(17)

The integral is trivial since the velocity is constant (Eqs. (16)), and we are ignoring the effects of actual energy change induced by this phase change. Ignoring motion in \( y \) and integrating over one focus (or defocus) section of length \( L_q \) yields

\[
| \Delta \psi | = \frac{k_t}{2} \frac{b_0^2}{y^3} \left( \frac{k_x}{k_u} \right)^2 \epsilon_n \bar{\beta} L_q
\]

(18)

where

\[
\bar{\beta} = \gamma \langle x_\beta^2 \rangle / \epsilon_n
\]

(19)

is the average focusing beta function and has units of inverse length. The phase change should be negligible, \( | \Delta \psi | \ll 2 \pi \), to minimize pernicious effects on the FEL action. Section 6 uses this result to evaluate the effectiveness of sextupole AG focusing.

In the smooth approximation, if the average beta function is the order of the gain length (\( \beta \sim L_g \)) then FEL operation (power output) is optimized [20]. Perturbations caused by the focusing on a scale longer than the gain length should not be significant. The next section examines an approximate solution to AG sextupole focusing and addresses the issue of focusing strength, to allow for a quantitative analysis of this scheme.
4. Matrix description of AG focusing

It is useful and straightforward to solve for the focusing effects of the AG sextupoles by using the transfer matrix description of the linear equations of motion. It is possible to use this method by substituting the equivalent quadrupole strength of the sextupole channel. This analysis will elucidate the effects of focusing on the beam size variation.

The transfer matrices for half of the focus (F) and defocus (D) section in a strong focusing lattice are defined by this prescription as follows

\[
F = \begin{bmatrix}
\cos \frac{\theta}{2} & \frac{1}{k_\beta} \sin \frac{\theta}{2} \\
-k_\beta \sin \frac{\theta}{2} & \cos \frac{\theta}{2}
\end{bmatrix},
\]

\[
D = \begin{bmatrix}
\cosh \frac{\theta}{2} & \frac{1}{k_\beta} \sinh \frac{\theta}{2} \\
k_\beta \sinh \frac{\theta}{2} & \cosh \frac{\theta}{2}
\end{bmatrix},
\]

respectively, where \( \theta = k_\beta L_q \) and \( L_q \) is the effective quadrupole length of each section. Then, the total transfer matrix for one cell (one period of the focusing channel) is given by

\[
M_1 = FDDF \quad (M_2 = DFFD),
\]

where a cell is started from the middle of a focus (defocus) section. Then,

\[
M_1 = \begin{bmatrix}
\cos \theta \cosh \theta & \frac{1}{k_\beta} (\sin \theta \cosh \theta + \sinh \theta) \\
-k_\beta (\sin \theta \cosh \theta - \sinh \theta) & \cos \theta \cosh \theta
\end{bmatrix},
\]

\[
M_2 = \begin{bmatrix}
\cos \theta \cosh \theta & \frac{1}{k_\beta} (\sin \theta + \sinh \theta \cos \theta) \\
-k_\beta (\sin \theta - \sinh \theta \cos \theta) & \cos \theta \cosh \theta
\end{bmatrix}.
\]

The parameter \( \mu \), the phase advance per cell, is then defined by \( 2 \cos \mu = \text{Tr}(M) = 2 \cos \theta \cosh \theta \). For small angles (\( \mu < \pi/4 \)) we may expand this transcendental expression to yield, in the smooth approximation,

\[
\mu \approx \theta^2/\sqrt{3}.
\]

The average beta function can be defined to be a geometric average of the minimum and maximum beta functions: \( \beta_{\text{min(max)}}(\mu) = [M_{1(2)}]_{1,2} \). This produces the relation

\[
\bar{\beta} = \frac{2\sqrt{3}}{k_\beta L_q}.
\]

We note at this point the strong dependence on \( k_\beta \), and that this beta function is \( \sqrt{3} \) times larger than that for a thin lens FODO channel. Although this implies a larger beam (and so less dense), the variation of the beam size is smaller than in the thin lens case. This is advantageous in an FEL since large fluctuations in the beam size may be deleterious to gain and optical beam quality. The above relation can be used to show that in a given FEL the phase change of Eq. (18) is approximately constant for small \( \mu \). The resulting relation can be written as

\[
|\Delta \psi| \approx 2\sqrt{3} \epsilon_n k_\gamma/\gamma.
\]

According to the FEL resonance condition \( k_\gamma \) scales as \( \gamma^2 \) implying that \( \Delta \psi \) scales with \( \gamma \). This indicates that high energy, short wavelength devices may be unable to use AG sextupoles. Note that the scaling of this result is expected from Eq. (3). In fact, the limit on the emittance difference only by a factor of \( \sqrt{3} \). Thus, Eq. (25) imposes a more stringent limit on the emittance than the 1D no focusing limit.

In order to make the strong focusing based on sextupole fields attractive, it must be clearly superior to weak focusing. This requires that

\[
\tilde{\beta} < \beta_{\text{weak}},
\]

where the quantity \( \beta_{\text{weak}} \) is defined as the beta function obtained for a round beam using Scharlemann’s pole shaping scheme. Let the ratio between the strong and weak betatron wavenumber be \( R = k_{\beta\text{strong}}/k_{\beta\text{weak}} \). Then \( k_{\beta\text{strong}} = R k_{\beta\text{natural}}/\sqrt{2} \) and Eq. (26) becomes [21]

\[
R > \left( \frac{2\sqrt{3}}{k_{\beta\text{weak}} L_q} \right)^{1/2} = \left( \frac{4\sqrt{3} \gamma}{b_0 L_q} \right)^{1/2}.
\]

This requirement is sometimes an overestimate of the ratio \( R \) because of the previous requirement that the smooth approximation be valid. To show what happens when this requirement is lifted, consider the case of 90° phase advance per cell (\( \mu = \pi/2 \)). Then,

\[
\tilde{\beta} = \frac{1}{k_\beta} \left( 1 - e^{-\pi/2} \right) \approx 2.2.
\]

Notice that the beta function in this case is independent of the quadrupole length (assuming \( L_q > \lambda_a \)). In fact, the ratio \( R \) is independent of all parameters and it is only required that

\[
R > 2.2.
\]

The variation of the undulator magnetic field by a strong sextupole component may be large enough to degrade the FEL synchronism condition, and this ef-
Fig. 1. Results of numerical simulations show the length of undulator required to reach saturation as a function of the strong sextupole focusing phase advance per cell. Large phase advances imply poor FEL performance.

The effect must be examined. The fractional variation of the magnetic field over the beam size can be expressed as

$$\left( \frac{\Delta B}{B} \right)_{\text{beam}} = \frac{k_s \sigma_x}{2} = \frac{(Rk_u \sigma_x)^2}{4},$$

(30)

where $\sigma_x$ is the transverse beam size. Similarly, the variation of the magnetic field over the undulation orbit is

$$\left( \frac{\Delta B}{B} \right)_{\text{undulator}} = \frac{1}{2} \left( \frac{Rb_n}{\gamma k_u} \right)^2.$$

(31)

As the examples in the last section will show, requiring that these variations be small compared to unity is not unreasonable. However, a large phase advance per cell can introduce problems. While a phase advance per cell of 90° minimizes the average beam envelope, it creates large fluctuations in the beam size. Numerical simulations [22] confirm that when the phase advance is large and hence the beam is modulated a great deal, the FEL action will be degraded (see Fig. 1). This statement also holds for quadrupole focusing [23]. Thus, in practice, the phase advance per cell must be smaller than 90°.

5. Implementing AG sextupole focusing

Sextupole focusing can be implemented by machining a parabolic curve into the permeable metal pole pieces of a hybrid undulator (see Fig. 2). It might also be possible to achieve an effective sextupole component by simpler methods. One scheme recently discussed is the use of side arrays of permanent magnets to shape the undulator field [24]. Another idea under consideration is the use of planar permanent magnets (see Fig. 3) [25].

6. Numerical examples

Three examples of high gain FEL amplifiers are now discussed to show when AG sextupole focusing in undulators is potentially useful. The proposed SLAC based X-ray FEL poses a number of challenges including the need to propagate the electron beam along ~ 50 m of undulator [26]. Table 2 presents the nominal beam and undulator parameters. The natural round beam focusing beta function is ~ 80 m whereas the design gain length requires a ~ 10 m beta function. So, weak focusing is insufficient to maintain the desired beam size and attain the design gain length.

We consider implementing strong focusing in this undulator. An average beta function of 7.8 m can be obtained with a focusing section length of 6 m and $R = 21.4$ when a 90° phase advance per cell lattice is used. Note that this section length is about equal to a

<table>
<thead>
<tr>
<th>Table 2</th>
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<tbody>
<tr>
<td><strong>The nominal beam and undulator parameters for the proposed SLAC X-ray FEL</strong></td>
</tr>
<tr>
<td><strong>Electron beam energy</strong></td>
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<tr>
<td><strong>Beam emittance (normalized, rms)</strong></td>
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<tr>
<td><strong>Peak current</strong></td>
</tr>
<tr>
<td><strong>Pulse length</strong></td>
</tr>
<tr>
<td><strong>Undulator magnetic field</strong></td>
</tr>
<tr>
<td><strong>Undulator period</strong></td>
</tr>
<tr>
<td><strong>Radiation wavelength</strong></td>
</tr>
<tr>
<td><strong>FEL parameter</strong></td>
</tr>
<tr>
<td><strong>AG phase variation $\Delta \psi$</strong></td>
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</table>
gain length, and thus the deleterious effects of the sudden change in z-velocity of the electrons at the transitions might be mitigated. It should be noted that in this example, the fractional variation of the magnetic field over the beam cross section is small ($\sim 2 \times 10^{-4}$), and the fractional variation of the undulator field over the undulating design orbit is even smaller ($\sim 4 \times 10^{-5}$). However, the large beam size variation would still be harmful. Thus, we examine focusing channels with smaller phase advances per cell. For $\mu \sim 10^7$, an average beta of $\sim 5$ m can be achieved with 0.5 m sections and $R = 70$. The phase variation of Eq. (25) is of the order of $2\pi$; strong sextupole focusing is not ideal for these parameters. It is useful to compare the results of numerical simulations with semi-analytic theories. Fig. 4 shows the results of such a comparison. As expected, the sextupole focusing shows poorer performance that the smooth approximation analytic theory. Both the theory and simulations attempt to account for 3D effects, but to simplify comparison the energy spread of the beam was taken to be zero.

We now examine an example with a much lower beam energy. The UCLA Particle Beam Physics Laboratory is constructing a 10.6 $\mu$m FEL for the purpose of studying physics issues relevant to future short wave-length operation including start up from noise (SASE), saturation and superradiance (see Table 3 for the design parameters) [27]. While the initial design calls for a single undulator section 60 cm long, future plans include adding a second section for a total length $\sim 120-160$ cm. The need for focusing might then become significant.

The short period modified hybrid undulator has flat poles. The natural vertical focusing has an equilibrium beta function of 10.5 cm, which if converted to the equivalent weak focusing round beam case would yield a function of 14.8 cm. While the phase variation (Eq. (25)) is small, strong focusing would yield only a modest improvement in the average beta function because of the short undulator period. The $\sim 9$ cm beta function attainable for reasonable phase advance per cell would only yield a modest increase in FEL performance.

The third example is based on the use of the Paladin undulator at SLAC [28]. This example uses a similar beam but with lower energy than the first example (see Table 4). The reduced beam energy decreases the phase variation across the focusing/de-
focusing boundaries enabling the AG sextupole focusing to approach the smooth approximation performance (see Fig. 5).

7. Conclusions

Alternating gradient sextupole undulator focusing has been examined for use in free electron lasers. Sextupole fields may be an attractive option for strong focusing provided that variation of the longitudinal velocity and transverse positions of the beam particles are minimized. The work presented here has arrived at three limits which an alternating gradient focusing scheme for an FEL must satisfy:

1) The velocity (or, equivalently, phase) modulation between focusing and defocusing sections must be small.
2) The beam size variation (or phase advance per cell) must be small.
3) The fractional variation of the magnetic field across the beam should be small.

Also, while not a fundamental limit, AG focusing should be stronger than weak (or natural) focusing to be practical.

It is the opinion of the authors that future FEL devices will rely on very strong (emittance limited) focusing. In light of this, more work is required on this subject. Certainly, field shaping alternatives need to be explored. Further work is also needed to examine the effects of AG sextupole focusing on FELs with various operating parameters. A more extensive comparison of both quadrupole and sextupole FEL focusing can help establish their relative merits. A consistent theory of FEL focusing, including betatron oscillations and beam size variations, is also lacking [29].

Acknowledgments

The authors are grateful to Claudio Pellegrinifor many useful conversations. We also thank Roman Tatchyn and Mark Hogan for their assistance. This work performed with partial support from U.S. Department of Energy grants DE-FG03-90ER40796 and DE-FG03-92ER40693, the SSC Fellowship program Texas National Research Laboratory grant FCFY9308, the Sloan Fellowship grant BR-3225, the Department of Education and the University of California.

References

[7] For simplicity we ignore the Bessel function coefficient 

\[ J' = J_0(x) - J_1(x) \text{ where } x = a^2_w/(1 + 2a^2_w) \].

[9] Here it is assumed that the electron and radiation beam sizes are equal. These relations hold for natural focusing and must, at least, be satisfied for other types of focusing. Another related condition that must be met is that the Rayleigh range be greater than the gain length, but this is not directly relevant to focusing.
[18] It is interesting to note that the problem possesses four length scales: the undulator period, the AG period, the betatron orbit, and the synchrotron period. We eliminate the undulator period by averaging the electron motion on this length scale. And, we neglect the synchrotron period since it is typically very long.
[21] Here it is assumed that the weak transverse focusing is equal in the two planes. That is, 

\[ k^2_2 = k^2_3 = k^2_4 / 2 \].

The relation \( \beta = 1/k' \) holds throughout.
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[26] H. Winick et al., 8th Nat. Conf. on Synchrotron Radiation, Gaithersburg, MD, USA (1993).


[29] Theories which include some focusing effects have been derived. See, for instance, Y.H. Chin, Proc. 14th Int. FEL Conf. Kobe, Japan (1992) and L.H. Yu, C.M. Hung, D. Li and S. Krinsky, Scaling function for free-electron-laser gun including alternating gradient focusing, to be published.