HAMILTONIAN MODEL OF A FREE ELECTRON LASER

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Both the Compton and the Raman regimes of a free electron laser are described by a relativistic hamiltonian which originates the evolution equations for 2N + 2 canonically conjugate electron and field variables, with the space coordinate as the independent variable. Space charge and field contribution to electron transverse velocity are included. Scaled variables are introduced which allow for a description of the behaviour of the system in terms of a single electron-beam parameter.

The free electron laser (FEL) is potentially an ideal tool for basic and applied research as a powerful source of tunable coherent radiation, generated via the injection of relativistic electrons in an undulator [1]. The coupled dynamics of the electrons and the radiation field is such that under proper conditions the radiation emitted by the accelerated particles can grow exponentially along the undulator. This high-gain regime is due to a collective instability of the system [2-5] and has been recently demonstrated both in the single-pass amplifier configuration [6] which we shall consider here, and even in the oscillator mode of operation [7].

The classical hamiltonian approach has been successfully used to describe the FEL process since the early days of single-electron, small-signal treatments [8]. We have applied this approach to the investigation of collective effects in the high-gain regime. In particular, a many-electron hamiltonian model for FEL amplifiers was discussed in ref. [3] in the limit of low density and negligible space-charge. Then in refs. [4,5] FEL dynamics was discussed dropping this restriction. In these papers some terms due to the radiation field contribution to electron transverse velocity were neglected; though small in many cases, they can become important for very long or tapered undulators. In this paper we present a set of evolution equations, valid both in the Compton and in the Raman regimes, which is more general than that of refs. [4,5] and preserves the hamiltonian structure of the system, so that energy conservation and Liouville theorem in a (2N + 2)-dimensional phase space are valid. A suitable scaling of variables allows for a description of FEL dynamics only in terms of one parameter.

The hamiltonian equations with time as the independent variable can be derived from the modified Hamilton principle [9]

$$\delta \int_{t_1}^{t_2} \left( p_x \frac{dx}{dt} + p_y \frac{dy}{dt} + p_z \frac{dz}{dt} - H \right) dt = 0. \tag{1}$$

Since in our problem one follows the evolution of the system along the undulator axis z, we change the independent variable from t to z, and using $H = E$ we obtain [10]

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In eq. (2) \((x, p_x), (y, p_y)\) and \((t, -E)\) appear as canonical variables with respect to a new hamiltonian \(H_1 = -p_z\). Hence we can write
\[
\frac{dx}{dz} = -\frac{\partial p_x}{\partial p_z}, \quad \frac{dy}{dz} = -\frac{\partial p_y}{\partial p_z}, \quad \frac{dp_x}{dz} = \frac{\partial p_x}{\partial x}, \quad \frac{dp_y}{dz} = \frac{\partial p_y}{\partial y},
\]
with
\[
\frac{dt}{dz} = \frac{\partial p_z}{\partial E}, \quad \frac{dE}{dz} = -\frac{\partial p_z}{\partial t}.
\]

Eqs. (4) are our working equations. Now let \(H\) be the relativistic hamiltonian for one electron interacting with electromagnetic fields
\[
H = c \left( \frac{p^2}{m_0 c^2} + eV \right) + eV = \gamma mc^2 + eV = E.
\]
In eq. (5) we assume that the vector potential \(A = A(z)\) is transverse and \(V = V(z)\) represents space-charge effects due to density fluctuations in an electron beam. Hence eqs. (4) become
\[
\frac{dt}{dz} = \left( \frac{1}{mc^2} \right) \frac{\partial p_z}{\partial \gamma},
\]
while from eq. (3) it follows that \(p_x = p_y = \text{const.}\), so that one can set \(p_x = p_y = 0\) thus obtaining from eq. (5)
\[
p_z = mc\sqrt{\gamma^2 - 1 - a^2}, \quad a = -\frac{eA}{mc^2}.
\]

By assuming that \(\gamma^2 \gg 1 + a^2\) we have
\[
p_z = mc\left( \gamma - \frac{(1 + a^2)^{1/2}}{2\gamma} \right).
\]

Now we specify the dimensionless vector potential \(a(z, t)\) as the sum of a magnetostatic, spatially periodic undulator potential \(a_0(z)\) and a radiation field potential \(a_L(z, t)\):
\[
a = a_0 + a_L, \quad a_0 = \left( \frac{\omega_0}{\omega_L} \right) \left[ e \exp(-ik_0z) + \text{c.c.} \right],
\]
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\]
where \(k_0 = 2\pi/\lambda_0 = \omega_0/c\) is the wavenumber associated with the undulator periodicity, \(k_L = 2\pi/\lambda_L = \omega_L/c\) is the radiation wavenumber, and circular polarization is taken for a helical undulator. Thus eq. (7') becomes
\[
p_z = mc\left( \gamma - \frac{(1/2\gamma)^{1/2}}{2\gamma} \left[ 1 + a_0^2 + ia_0 \left( a_{L*} \exp(-i\theta) - \text{c.c.} \right) + |a_L|^2 \right] \right),
\]
where \(\theta\) is the electron-field phase
\[
\theta = (k_L + k_0)z - \omega_L t.
\]

As \(d\theta/dz = k_L + k_0 - \omega_L dt/dz\), the evolution equations for the variables \(\theta\) and \(\gamma\) can be obtained at once from (6), (6') and (9); or, alternatively, via a canonical transformation from \((t, -E)\) to \((\theta, \gamma)\), as well known in accelerator physics [10].

Without any further approximation we get
\[
\frac{d\theta_j}{dz} = k_L \left[ 1 - \gamma_j^2 - \frac{\gamma_j^2}{2\gamma_j^2} \left( a_0(a_{L*} \exp(i\theta_j - \text{c.c.}) - |a_L|^2) \right) \right],
\]
\[
\frac{d\gamma_j}{dz} = -k_L \left[ \frac{1}{2} a_0(a_{L*} \exp(i\theta_j + \text{c.c.}) + i(\omega_p/\omega_L) \exp(-i\theta_j) \exp(i\theta_j - \text{c.c.}) \right].
\]

where \(\omega_p = (4\pi e^2 n/mc^2)\) is the plasma frequency, \(n = N/V\) the electron number density, \(\gamma_R = [\omega_L (1 + a_0^2)/2 \omega_0]^{1/2}\) the resonance energy (at zero initial field) in rest energy units, \(\exp(-i\theta) = N^{-1} \sum_{j=1}^N \exp(-i\theta_j)\) the electron
bunching parameter. In eqs. (10), (11) we have added an index-4 to distinguish different electrons in a beam. Furthermore, the space-charge contribution to $d\gamma/dz$ is obtained from eq. (6') using the result of ref. [5] for the first harmonic contribution to $E_z$,

$$
\langle E_z \rangle = -(i 4\pi e n/k_L) \left[ \exp(-i\theta) \exp(i\theta) - c.c. \right].
$$

The field equation can be derived as in ref. [4] from Maxwell equations by neglecting slippage effects, and reads

$$
da_L/dz = \left( \sigma / (2c) \right) \left( \langle \psi_0 \rangle \langle \psi \rangle \right) - i (1/\gamma) a_L,
$$

where again any bracket $\langle \rangle$ means an average $N^{-1} \sum_{j=1}^{N}$. The last term in eq. (12), which is usually neglected, comes from the radiation field contribution to electron transverse velocity, just like the terms depending on the field in eq. (10). However, we stress that it is not necessary to derive eq. (12) as usual from Maxwell equations since, as we demonstrate here, eqs. (10), (11) and (12) can be obtained as hamilton equations from a unique hamiltonian for both electron and field variables.

Actually, by introducing the scaling as in ref. [4]:

$$
\tilde{\gamma}_j = \gamma_j - \delta \tilde{z}, \quad \Gamma_j = (1/\rho)(\gamma_j/\gamma_0), \quad \tilde{z} = 2k_0(\gamma_R^2/(\gamma_0^2)) \rho z,
$$
eqs. (10)–(12) become

$$
d\tilde{\psi}_j/d\tilde{z} = (1/\rho) \left( 1 - \frac{1}{\rho^2} \right) + i \left( \frac{1}{\rho} \right) \left( A \exp(i\psi_j)/\Gamma_j^2 - c.c. \right) - \frac{i}{2} \sigma \tilde{A}_j^2/\Gamma_j^2 \quad (j = 1, \ldots, N)
$$

$$
d\Gamma_j/d\tilde{z} = - \frac{1}{\rho} \left( A \exp(i\psi_j)/\Gamma_j + c.c. \right) - i \sigma \left( \exp(-i\psi_j) \exp(i\psi_j) - c.c. \right) \quad (j = 1, \ldots, N)
$$

$$
dA/d\tilde{z} = \left( \frac{1}{\rho} \right) \left( \exp(-i\psi_j)/\Gamma_j + i(\delta - \frac{1}{2} \sigma \langle 1/\Gamma_j \rangle) A \right),
$$

where the generalized Pierce parameter $\rho$, the detuning parameter (at zero initial field) $\delta$ and the space-charge parameter $\sigma$ are defined as

$$
\rho = (1/\gamma_0) \left( \frac{1}{\rho} \right) \left( \gamma_0^2/\gamma_R^2 \right) \omega_p/\omega_0)^{2/3}, \quad \delta = (1/2\rho) \left( \gamma_0^2 - \gamma_R^2 \right)/\gamma_R^2, \quad \sigma = 4\rho(1 + a_0^2a_0^2).
$$

Notice that the last terms of eqs. (14a) and (14c), which are due to the transverse velocity induced by the radiation field, have the same coefficient $\sigma$ of the space-charge term. Since $\sigma$ depends essentially on $\rho$, the whole system of evolution equations (14) is ruled only by the parameter $\rho$.

Eq. (14a–c) can be obtained immediately from the following hamiltonian

$$
\bar{H} = \frac{1}{2\rho} \sum_{j=1}^{N} \left( \Gamma_j + 1/\rho^2 \Gamma_j^2 \right) + i \rho \left( A^* \sum_{j=1}^{N} \exp(-i\psi_j) / \Gamma_j - c.c. \right) - \left( \delta - \frac{1}{2} \sigma \langle 1/\Gamma_j \rangle \right) N|\tilde{A}|^2 + \sigma N|\exp(-i\psi_j)|^2
$$

$$
= \frac{1}{2\rho} \sum_{j=1}^{N} \left( \Gamma_j + 1/\rho^2 \Gamma_j^2 \right) + (1/\rho)(2/N)^{1/2} \left( \Gamma_0 \sum_{j=1}^{N} \cos \psi_j / \Gamma_j + \psi_0 \sum_{j=1}^{N} \sin \psi_j / \Gamma_j \right)
$$

$$
- \frac{1}{2} \left( \delta - \frac{1}{2} \sigma \langle 1/\Gamma_j \rangle \right) \left( \psi_0^2 + \Gamma_0^2 \right) + \sigma \langle \exp(-i\psi_j) \rangle^2,
$$

where

$$
A \equiv \left( \psi_0 + i\Gamma_0 \right)/(2N)^{1/2},
$$
as the canonical equations

$$
d\tilde{\psi}_j/d\tilde{z} = \partial \bar{H}/\partial \Gamma_j, \quad d\Gamma_j/d\tilde{z} = - \partial \bar{H}/\partial \psi_j \quad (j = 0, 1, \ldots, N).
$$

Note that when $j = 0$ in eq. (18), the two real equations can be combined into $dA/d\tilde{z} = -(iN)^{-1}(\partial \bar{H}/\partial A^*)$, which leads directly to eq. (14c).
Eqs. (14) admit another constant of motion besides (16), namely
\[ \langle \Gamma \rangle + |A|^2 = \text{const.} \]  
(19)
which rules the global energy exchange between the electrons and the field.

Eqs. (14) generalize those of ref. [5], where also harmonics were considered for a planar undulator, since they include all contributions due to the effect of the radiation field on the electron transverse velocity. Dropping these terms and the space-charge terms as well one obtains the equations of ref. [4]. In the limit
\[ \rho n_j = \rho \gamma_j - 1 = (\gamma_j - \langle \gamma \rangle_0) / (\gamma_0) \ll 1, \]  
(20)
which is valid up to \( \rho \ll 0.1 \), one can keep only first-order contributions in the relative energy variations (20). By taking only the zeroth-order contribution in the tiny term \( \propto |A|^2 \) one obtains from (16) a hamiltonian,
\[ \tilde{H}_1 = \frac{1}{2} \sum_{j=1}^{N} \eta_j^2 + i \left( A^* \sum_{j=1}^{N} \exp(-i\psi_j) (1 - \rho \eta_j) - \text{c.c.} \right) - \delta_1 N|A|^2 + \sigma N (\exp(-i\psi)|^2, \]  
(21)
which still fully describes the exponential gain regime, as we shall see below. On a low density, \( \rho \ll 1 \), one can neglect the first-order corrections in the interaction term of eq. (21). Space charge can be neglected as well, provided that \( \sigma \ll 1 \). In these limits hamiltonian (21) is approximated by [3, 11]
\[ \tilde{H}_0 = \frac{1}{2} \sum_{j=1}^{N} \eta_j^2 + i \left( A^* \sum_{j=1}^{N} \exp(-i\psi_j) - \text{c.c.} \right) - \delta N|A|^2. \]  
(22)
recently considered in investigating the existence of a collective variable description and the occurrence of optical guiding in high-gain amplifiers [12]. Notice that no beam parameter appears in eq. (22), which then can be used as a universal hamiltonian for a large class of free electron lasers.

Hamiltonian (16) can be considered exact in a one-dimensional classical treatment of constant-wiggler FEL dynamics, where only transverse effects (to be simulated via an effective energy spread) and diffraction (counteracted by optical guiding in the high-gain regime) are not included. It can be easily generalized to many radiation modes and also extended to harmonics in the case of a planar undulator. Note that equations equivalent to eqs. (14) were already considered [10, 13, 14], but never in the framework of a fully hamiltonian treatment for both many electrons and the field. Furthermore, the scaling adopted here is most suitable to discuss the collective instability and the high-gain regime.

Next, we focus on the collective instability and exponential gain regime. Eqs. (14) admit a stationary solution
\[ A_0 = 0, \quad \langle \Gamma \rangle_0 = 1/\rho, \quad \sum_{j=1}^{N} \exp(-in\psi_j^0) = 0, \]  
(23)
i.e., in the absence of field excitation and for a monoenergetic and unbunched electron beam. The linear stability analysis around condition (23), performed e.g. by the technique of ref. [4], leads to a cubic characteristic equation for the parameter \( \lambda \), where \( A(\delta) \propto \exp(i\lambda \delta) \), which can be written as a travelling-wave-tube dispersion relation
\[ (\lambda^2 - \sigma) (\lambda - \delta_1) + 2\rho \lambda + 1 + \rho^2 \sigma = 0. \]  
(24)
This cubic equation is associated with hamiltonian (21), which can thus replace the full hamiltonian (16) as far as only linear amplification is considered. The instability condition for the cubic equation (24) is
\[ 27(1 + \sigma_1)^2 - 4[\rho_1^3 + (1 + \sigma_1) \delta_1^2] - \rho_1^2 \delta_1^2 - 18(1 + \sigma_1) \rho_1 \delta_1 > 0, \quad \rho_1 = \sigma - 2\rho, \quad \sigma_1 = \sigma(\delta_1 + \rho^2). \]  
(25)
However, we do not further discuss the stability of the full dispersion relation (24) because similar equations were extensively studied since the late forties in connection with microwave electronics [15]. On the contrary we consider two limit cases, corresponding to the Compton and the Raman FEL regimes [1, 2, 14].

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(i) In the limit $\rho \to 0$ of low density and negligible space charge, eq. (24) reduces to the cubic equation

$$\lambda^3 - \delta \lambda^2 + 1 = 0$$

(26)

associated with the reduced Hamiltonian (22). In this case the system is unstable when $\delta \leq \delta_T = (27/4)^{1/3} \approx 1.89$ [4]; this instability leads to self-bunching of the electrons and exponential growth of radiated intensity before saturation sets in, that is, *self-amplified spontaneous emission*, a most promising mechanism to generate coherent radiation up to the XUV region [16]. The peak power turns out to scale as $N^{4/3}$, thus showing the cooperation among the electrons via the common radiation field. The maximum spatial growth rate ($\propto \rho$) occurs on resonance, $\delta = 0$, with a spectrum peaked at the resonance frequency $\omega = 2\omega_0 \gamma_R^2/(1 + \alpha_0^2)$. In a reference frame moving at the mean electron velocity the process can be described as a stimulated Compton backscattering of the undulator wave.

(ii) If one keeps terms up to order $O(\rho)$, so that space charge cannot be neglected, the dispersion relation (24) is approximated by

$$\lambda^3 - \delta \lambda^2 - (\alpha - 2\rho) \lambda + 1 + \alpha \delta = 0.$$  

(27)

Due to high-density effects the system changes its operation regime from Compton to Raman, which is collective in a stronger sense due to the occurrence of plasma oscillations. Also, the instability domain is limited with respect to $\delta$ and shrinks as space-charge forces increase. The maximum growth rate ($\propto \rho^{3/4}$) shifts from $\delta = 0$ to $\delta = \alpha^{1/2}$ and accordingly the operation frequency becomes $\omega = [2(\gamma_0^2/(1 + \alpha_0^2)) [\omega_0 - (\alpha_0^2/(1 + \alpha_0^2))] |^{1/2} \gamma_0^2]^{1/2}$ [1,2,5]. In the moving frame the process appears as a stimulated Raman backscattering in which the laser (signal) frequency is equal to the difference between the undulator (pump) and the plasma (idler) frequencies.

Our discussion is supported by numerical results obtained from the Hamilton equations (14) and the associated dispersion relation (24).

Fig. 1 shows the imaginary part, $\text{Im} \lambda$, of the root of the cubic eq. (24) which rules the exponential amplification, plotted as a function of detuning $\delta$ for different values of the parameter $\rho$ ranging from $\rho = 10^{-3}$ to $\rho = 1$. For $\rho \ll 0.01$ (figs. 1(a), (b)) the behaviour is typical of the two-wave Compton instability, whereas for $\rho = 0.1$ (fig. 1(c)) the behaviour is that of the three-wave Raman instability, as discussed above. At $\rho = 1$ (fig. 1(d)) the maximum gain is at larger $\delta$ and is also raised; however, such a high value of $\rho$ is meaningful only provided that the gain over one radiation wavelength is small, which sets a restriction $\lambda_{\text{L}} k_0 \rho^{3/4} \ll 1$ in the Raman regime.

In figs. 2, 3 we show the evolution of radiated intensity along the undulator axis as calculated from the Hamilton eqs. (14) in the case of the Compton (fig. 2) and Raman (fig. 3) instabilities. In both cases the system starts from noise close to equilibrium condition (23) and with maximum spatial growth rate as determined from figs. 1(b), (c), i.e., $\rho = 0.01$, $\delta = 0.09$ in fig. 2 and $\rho = 0.1$, $\delta = 0.78$ in fig. 3. One sees the same qualitative behaviour in the two intensity patterns. Notice that the first peak in the Raman regime (fig. 3) is higher (and slightly delayed) with respect to the first peak in the Compton regime (fig. 2), which is due to high density effects. Hence the energy transfer from the particles to the field is more efficient in the Raman than in the Compton FEL, though they operate in different domains of the spectrum [1].

In conclusion, both the Compton and the Raman FEL regimes are fully described by our Hamiltonian treatment, which includes space charge and the contribution to transverse electron dynamics induced by the radiation field, and depends essentially on a single parameter.
Fig. 2. Compton regime: Field intensity $|A|^2$ versus longitudinal coordinate $z$ calculated from eqs. (14), with initial conditions $A_0 = 0$, $(1/\rho) \rho_0 = \exp(-i\psi) \approx 0.15$. The parameters are: $\rho = 0.01$, $\delta = 0.09$, $\omega_0 = 1$.

Fig. 3. Raman regime: Field intensity $|A|^2$ versus longitudinal coordinate $z$ calculated as in fig. 2, but with: $\rho = 0.1$, $\delta = 0.78$.

References

[1] See e.g. T.C. Marshall, Free-electron lasers (Macmillan, New York, 1985);
   Free electron lasers, eds. J.M.J. Madey and A. Renieri (North-Holland, Amsterdam, 1985) (Nucl. Instr. and Meth. A 237 (1,2));
   Coherent and collective properties in the interaction of relativistic electrons and electromagnetic radiation, eds. R. Bonifacio, F. Casagrande and C. Pellegrini (North-Holland, Amsterdam, 1985) (Nucl. Instr. and Meth. A 239 (1)).

   G. Dattoli, A. Marino, A. Renieri and F. Romanelli, Electron. QE-17 (1981) 1371;


   Rev. Lett. 48 (1982) 238;


   and electromagnetic radiation,, eds. R. Bonifacio, F. Casagrande and C. Pellegrini (North-Holland, Amsterdam, 1985) (Nucl.
