COLLECTIVE INSTABILITIES AND HIGH-GAIN REGIME IN A FREE ELECTRON LASER

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We study the behavior of a free electron laser in the high gain regime, and the conditions for the emergence of a collective instability in the electron beam-undulator-field system. Our equations, in the appropriate limit, yield the traditional small gain formula. In the nonlinear regime, numerical solutions of the coupled equations of motion support the correctness of our proposed empirical estimator for the build-up time of the pulses, and indicate the existence of optimum parameters for the production of high peak-power radiation.

Studies of the free electron laser (FEL) in the high gain regime have shown that with an appropriate selection of the electron density, detuning and undulator length, the radiation field and the electron bunching can undergo exponential growth as a result of a collective instability of the electron beam-undulator-radiation field system [1–8]. In this paper, we study the conditions for the onset of this instability using a new secular equation for the characteristic complex frequencies of the FEL system. On the basis of these results, we show how one can re-derive the small-signal gain formula and establish the conditions for its validity. We also consider the problem of the initiation of laser action and of the growth of the radiation field from noise, and propose a formula to evaluate the lethargy (build-up) time of the first pulse. Finally, we study the nonlinear regime of the FEL by numerical methods and obtain results that suggest the existence of an optimum efficiency of the device.

In the derivation of our working equations we select the phase and the energy as the basic electron variables, and assume the slowly varying phase and amplitude approximation for the radiation field as done also in earlier developments [9,10]. In the remainder of the paper we shall adopt the following notations: \( z \) represents the direction of propagation of the helical magnetic field and \( \lambda_0 \) and \( N_0 \) the period length and the number of periods of the undulator, respectively; the undulator parameter is \( \kappa = eB_0\lambda_0/(2\pi mc^2) \), where \( mc^2 \) is the electron rest energy; \( \lambda \) is the wavelength of the radiation field, \( \gamma \) is the electron energy in units of \( mc^2 \), \( \beta_z \approx 1 \) is the longitudinal electron velocity and \( \beta_\perp = \kappa/\gamma \) the amplitude of the transverse velocity; the electron phase, \( \phi \), relative to that of the electromagnetic wave, is connected to \( z \) and \( t \) by the relation \( \phi = 2\pi z/\lambda_0 + 2\pi(z - c t)/\lambda \); the resonant energy \( \gamma_R \) is related to \( \lambda_0 \) and \( \kappa \) by \( \gamma_R^2 = \lambda_0(1 + \kappa^2)/2\lambda \), and, finally, the undulator frequency \( \omega_0 \) is given by \( \omega_0 = 2\pi c\beta_\perp/\lambda_0 \).

With these notations, the FEL working equations can be written as [9,10]

\[ \dot{\phi}_j = \omega_0(1 - \gamma_R^2/\gamma_j^2), \]

\[ \gamma_j = -\frac{e\kappa}{2mc^2\gamma_j}\left[\alpha \exp(\imath\phi_j) + \text{c.c.}\right], \]
\[
\left( \frac{\partial}{\partial z} + \frac{1}{c} \frac{\partial}{\partial t} \right) \alpha = 2 \pi n_e \frac{K}{\Sigma} \langle e^{-i\phi}/\gamma \rangle ,
\]

where \( j \) labels the \( j \)th electron in the beam \((j = 1, 2, \ldots)\), \( N_e \) is the total number of electrons; the average \( \langle \cdot \rangle \) is carried out over all electrons in a beam slice of thickness \( \lambda \) at the position \( z = \langle \beta_z \rangle c t \), where \( \langle \beta_z \rangle \) is the average longitudinal velocity. The remaining parameters have the following meaning: \( n_e \) is the electron beam longitudinal density at position \( z = \langle \beta_z \rangle c t \), \( F \), is an effective beam transverse cross section describing the overlap of the beam with the radiation field whose amplitude \( E_0 \) and phase \( \phi_0 \) have been combined in the complex amplitude \( \alpha = i E_0 \exp(i \phi_0) \).

It is important to stress that in this discussion \( \gamma \) is not restricted to be approximately equal to the resonant value \( \gamma_R \), unlike earlier treatments of this problem.

For the purpose of our subsequent analysis, it is convenient to rewrite eqs. (1.3) using the variables

\[
z' = z - \langle \beta_z \rangle c t , \quad t' = t ,
\]

with the result:

\[
(\partial/\partial t') \phi_j = \omega_0 (1 - \gamma_R^2/\gamma_j^2) ,
\]

\[
(\partial/\partial t') \gamma_j = \frac{e c \varepsilon_0}{2 m c^2 \gamma_j} [\alpha \exp(-i \phi_j) + c.c.] ,
\]

\[
\left[ (1 - \langle \beta_z \rangle) \frac{\partial}{\partial z'} + \frac{1}{c} \frac{\partial}{\partial t'} \right] \alpha = 2 \pi n_e \langle \gamma' \rangle \frac{K}{\Sigma} \langle e^{-i\phi}/\gamma \rangle z' .
\]

The propagation term \((1 - \langle \beta_z \rangle) \partial/\partial z'\) in eq. (7) is important to describe the evolution of the pulse in the FEL, especially when the accumulated path difference \(\Delta L \equiv L_{\text{ph}} - L_{\text{el}} = (c - v) t_{\text{init}}\) between the photons and the electrons during an interaction time is comparable to the length of the electron bunch itself.

Note that in terms of eqs. (13)–(15), the dynamics of the FEL is controlled by only two parameters, the Pierce parameter \( \rho \) (eq. (9)) and \( \delta = \Delta/\rho \), where \( \Delta \) is the usual detuning \((\gamma_0^2 - \gamma_R^2)/(2 \gamma_R^2) \).

Because we neglect space-charge forces, we shall assume in the following that \( \rho \) is sufficiently smaller than unity. It is also worth noting that eqs. (13)–(15) are consistent with the conservation law

\[
L = |A|^2 + \langle \Gamma \rangle = \text{constant} ,
\]

or also

\[
L = mc^2 n_0 (\gamma) + E_0^2/4\pi = \text{constant} ,
\]

which can be readily recognized as the conservation of energy for the electron beam–radiation field system. The method devised to analyze the stability of the system is based on the procedure suggested in ref. [8]. The equations are linearized around the equilibrium state \( A_0 = 0, \Gamma_0 j = 1/\rho, \exp(-i \psi_0) = 0 \) and perturbed by letting \( A = a, \Gamma_j = (1/\rho)(1 + \eta_j) \) and \( \psi_j = \psi_0 j + \delta \psi_j \).

The linearized equations form the basis for a closed form linear system of equations for the collective variables

\[
x = \langle \delta \psi \exp(-i \psi_0) \rangle ,
\]

where \( \gamma_0 \) is the initial energy, and \( \rho_e \) the classical electron radius, and the so-called Pierce parameter

\[
\rho = (4 \pi \kappa (\gamma_0/\gamma_R)^2 \Omega_p/\omega_0)^{2/3} .
\]

Furthermore, we introduce the quantity

\[
\phi_0 = \omega_0 (1 - \gamma_R^2/\gamma_0^2) ,
\]

and rescale the time variable as follows:

\[
\tau = 2 \omega_0 \rho (\gamma_R/\gamma_0)^2 t .
\]

In terms of the new scaled variables

\[
\psi_j \equiv \phi_j - \phi_0 t , \quad \Gamma_j \equiv \gamma_j / (\rho \gamma_0) ,
\]

\[
A \equiv \alpha \exp(i \phi_0)/((4 \pi m c^2 \gamma_0 n_0 \rho^2)^{1/2} \exp(\psi_0) + c.c.] ,
\]

\[
(\partial/\partial \tau) \psi_j = (1/\rho^2) (1 - 1/\rho^2 \Gamma_j^2) ,
\]

\[
(\partial/\partial \tau) \Gamma_j = -(1/\rho)[(A/\Gamma_j) \exp(i \psi_j) + c.c.] ,
\]

\[
dA/\partial \tau = i \delta A + (1/\rho)(e^{-i \psi}/\Gamma) .
\]
\[ y = \frac{1}{p} (\eta \exp(-i\psi_0)) , \]  \hspace{1cm} (18)

and for the field perturbation \( \alpha \). These take the form

\[ \frac{dx}{dr} = y \]  \hspace{1cm} (19)

\[ \frac{dy}{dr} = -\alpha \]  \hspace{1cm} (20)

\[ \frac{d\alpha}{d\tau} = -i\delta \alpha - ix - \rho y . \]  \hspace{1cm} (21)

Nontrivial solutions with a time dependence of type \( \exp(i\lambda \tau) \) exist if and only if \( \lambda \) is a solution of the characteristic equation

\[ \lambda^3 - \delta \lambda^2 + \rho \lambda + 1 = 0 . \]  \hspace{1cm} (22)

The results of earlier analyses \([1-8]\) can be recovered by setting formally \( \rho = 0 \) in eq. (22). Clearly, exponential growths, and thus, unstable behavior, results if the cubic equation (22) has one real and two complex conjugate roots. In this case, the imaginary part of the eigenvalue measured the rate of growth of the unstable solution. The instability condition can be easily derived from eq. (22): in terms of the parameters \( \rho \) and \( \delta \) it takes the form (fig. 1)

\[ \rho^3 - \frac{1}{5} \rho^2 \delta^2 + \frac{9}{2} \rho \delta - \delta + \frac{27}{4} \geq 0 . \]  \hspace{1cm} (23)

The typical behavior of the eigenvalues of eq. (22) as a function of detuning is shown in fig. 2. The eigenvalues are real when \( \delta \) exceeds a certain threshold value that depends on \( \rho \) according to eq. (23), while two of the eigenvalues form a complex conjugate pair when \( \delta < \delta_{\text{thr}} \).

The small signal gain formula emerges in a natural way from our analysis in the limit \( \rho \to 0 \), and for sufficiently large values of \( |\delta| \). In this limit, the eigenvalues take the approximate form

\[ \lambda_1 \simeq \delta (1 - 1/\delta), \quad \lambda_{2,3} \simeq \pm 1/\delta^{1/2}, \quad \delta > 0 , \]  \hspace{1cm} (24)

\[ \lambda_1 \simeq \delta (1 - 1/\delta), \quad \lambda_{2,3} \simeq \pm 1/|\delta|^{1/2}, \quad \delta < 0 , \]  \hspace{1cm} (24)

as one can confirm qualitatively from fig. 2. The output field \( A(\tau) \) in the linear regime can be calculated as a linear superposition of elementary exponential functions whose coefficients are to be fixed from the initial conditions. A lengthy, but straightforward calculation yields the following expressions for the small signal gain:

\[ G \equiv \frac{|A(\tau)|^2 - |A_0|^2}{|A_0|^2} = \frac{4/\delta^3 (1 - \cos \delta \tau \cos \tau/\sqrt{\delta})}{\delta^3/2 \sin \delta \tau \sin \tau/\sqrt{\delta}} , \quad \delta > 0 , \]
\[ G = \frac{4}{63} \left( 1 - \cos \delta \cosh \frac{\tau}{\sqrt{\delta}} \right) - \frac{1}{2} \sqrt{\delta}^{3/2} \sin \delta \sinh \frac{\tau}{\sqrt{\delta}}, \quad \delta < 0. \] (25)

In order to make contact with the usual small-signal gain formula, it is not enough to require that \( |\delta| \) be sufficiently larger than unity, but one also must impose the condition \( \tau/\sqrt{|\delta|} \ll 1 \). In this case, eq. (25) becomes

\[ G \approx \frac{4}{63} \left( 1 - \cos \delta \tau - \frac{1}{2} \delta \tau \sin \delta \tau \right) \] (26)

which, in fact, agrees with the standard expression for \( G \).

In spite of the fact that the equations of motion of the FEL are nonlinear, some aspects of this problem can be handled accurately by analytic means. The evolution below threshold (\( \delta > \delta_{\text{thr}} \)) is governed by the linear approximation. In this regime, the eigenvalues are real (see fig. 2) and the output field displays small amplitude oscillations when plotted as a function of time. On varying \( \delta \), beat patterns or more complicated-looking modulation effects can be observed, whose origin can be understood entirely in terms of the eigenvalues of the linearized problem. A representative example is shown in fig. 3. It may be worth mentioning that while the trace in fig. 3 has been obtained by the appropriate superposition of exponential functions, the exact solution of eqs. (13)–(15) is indistinguishable on the scale of this graph.

The system evolution above threshold (\( \delta < \delta_{\text{thr}} \)) is entirely different, and is shown in fig. 4 for the case of zero initial field and an initial bunching parameter \( |\exp(-i\psi_0)| \), small, but different from zero. Under unstable conditions, fluctuations in the electrons injection velocities, or the lack of uniformity in the initial distribution of the electron phase variables, or the presence of an initial field will trigger the growth of a signal. The signal will then grow to a peak value after which it oscillates. This behavior is very general and is independent of the initial triggering mechanism as long as this perturbation is small. This nonlinear regime requires numerical integration of the full equations of motion. This we have done for a number of values of \( \rho \) and \( \delta \).

Because of the nature of the triggering mechanism, intuitively, one would expect that the time required for the initial pulse to build up (lethargy time) should be a fairly sensitive function of the magnitude of the initial fluctuation. We have examined the dependence of the build up time of the first pulse on the initial value of the bunching parameter, and verified that (a) a significant fraction of the build up process is well described by the linearized equations; and (b) the arrival time of the first peak is well described by the formula:

\[ \tau_{\text{peak}} = -(1/\text{Im} \lambda) \ln |\exp(-i\psi_0)| + 1. \] (27)

A test of this equation is provided in fig. 5, where we have plotted the arrival time of the first pulse calculated from the nonlinear equations of motion (13)–(15), as a function of the initial bunching parameter \( |\exp(-i\psi_0)| \). One aspect of considerable interest for
the purpose of optimizing the system's parameters is the existence of a maximum peak power output as a function of $\rho$ and $\delta$. We have verified that while the maximum growth rate is obtained for $\delta \approx 0$, the maximum peak amplitude of the first pulse occurs for $\delta \approx \delta_{th}$. Thus, we have scanned the $(\rho, \delta)$ plane in the neighborhood of, but above, the threshold line and for $A_0 = 0$, and recorded the peak output intensity $\rho|A|_{\text{max}}^2$ as a function of $\rho$ (fig. 6). Notice that it follows from eq. (16) that $\rho|A|^2 = (T_f - T_0)/\gamma_0$, so that $\rho|A|^2$ gives the energy transfer from the electrons to the radiation. The scatter of the points is almost certainly due to the slight variations of the conditions from run to run. The solid line, which is only a qualitative average through the points, suggests the existence of an optimum gain-detuning condition such that the efficiency of the system is maximum for operation just above threshold. It is clear that in the presence of efficiencies as large as, in principle, 40%, the old approximate treatments [1–8] in which the electron momentum is assumed to vary only by small amounts cannot be adequate to describe situations where such large energy exchanges take place between the electron beam and field. On the other hand, it is intuitively obvious that for sufficiently small values of the Pierce parameter, the electron energy will suffer only a limited depletion so that earlier treatments should be sufficiently accurate.

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