Coherent transition radiation from a helically microbunched electron beam

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The coherent transition radiation emitted from an electron beam with higher-order spatial microbunching is analyzed. The characteristic angular and phase dependence can be used to identify the dominant bunching structure of such beams, which can be generated during the harmonic interaction in optical klystron modulators and free-electron lasers, and used as tunable sources of coherent light with orbital angular momentum. © 2009 American Institute of Physics.

[I. INTRODUCTION]

Transition radiation (TR) occurs when a charged particle crosses a boundary separating two media with differing electromagnetic properties. The photon emission can be enhanced coherently at frequencies for which the emission from collections of charged particles is in phase. Distributions of longitudinally microbunched electrons, such as those generated in a free-electron laser (FEL) interaction, for example, emit coherently at the spatial bunching frequency, allowing coherent TR (CTR) to be used regularly as a microbunching diagnostic.2–6

Recently, the generation of electron beams (e-beams) with more complex helical microbunching structures (e.g., springlike density distributions) has been examined in the context of e-beam modulators that employ the inverse FEL mechanism.7 These higher-order correlated microstructures result naturally at higher frequency harmonics of the resonant interaction in a helical magnetic undulator between the e-beam and the input laser field. They also can occur, in a complementary way, when the e-beam interacts with optical modes in an FEL that possess a helical phase dependence,8 or in any such designed scenario where a resonant interaction is possible between the e-beam and a slow wave with this novel phase structure. The three-dimensional (3D) geometry of the resonant ponderomotive phase bucket is imprinted on the e-beam, resulting in density and velocity modulations with a corkscrewlike transverse and longitudinal correlation. Beams that are helically microbunched can also radiate coherently to generate coherent optical modes with a helical azimuthal phase, either in the FEL scenario mentioned above, or through other radiation processes such as CTR, though the radiation distribution is weighted by the differing attributes of the radiating mechanism. Either way, the emitted radiation fields are described by a superposition of optical modes that carry nonzero total orbital angular momentum (OAM).9 Further, since helical microbunching can conceivably be generated on x-ray length scales in the manner of the forward opening angle, and attosecond time scales attainable in modern x-ray light sources. It is useful, therefore, to examine the characteristic CTR emission from helically microbunched beams that can generate such short-wavelength OAM modes, in order that the CTR be used efficiently either as a microbunching diagnostic or as an experimental tool with unique attributes and applications.

Here, we investigate the classical spectral and angular characteristics of CTR emitted from a relativistic e-beam, with particular emphasis on beams that have a dominant helical microbunching geometry. The beam is assumed to strike a perfectly conducting infinite surface, and the radiation is assumed to be measured in a region far beyond the formation zone. This classical analysis is an extension of a previous framework which focused on simpler bunching geometries (i.e., purely longitudinal bunching10), and uses an expansion of the e-beam density modulation in terms of spatial modes. This approach reveals the subtle microbunching signatures that are imprinted on the emission from a simple beam with substantial symmetry, but can also be used to examine the cause of more complex transverse intensity structures.11

II. TR FROM AN IDEAL MIRROR

The radiation emitted from a single electron initially moving at constant velocity in vacuum which strikes a transversely infinite, perfect conductor has been examined extensively.12 The spectral radiation field energy emitted into the solid angle \(d\Omega\) is

\[
\frac{d^2U}{dkd\Omega} = \frac{e^2}{4\pi\varepsilon_0} \frac{\sin^2 \theta}{(1 - \beta^2 \cos^2 \theta)^2},
\]

where \(e=|q|\) is the electron charge, \(k=\omega/c\) is the wavenumber of the radiation and \(\beta c\) is the velocity (assumed relativistic so \(\beta=1\)). The solid angle is \(d\Omega=\sin \theta d\theta d\phi\), where \(\theta\) is the forward opening angle, and \(\phi\) is the azimuthal angle. Equation (1) illustrates two signature features of TR: the zero on-axis emission \((\theta=0)\) and the flat frequency response.

A. Coherent bunch TR

For multiple electrons \(N_e\), the TR distribution is calculated by summing over all the positions, velocities and arrival times of the electrons impacting the mirror. Here we
consider the continuous beam limit \((N_b \gg 1)\) for a cold beam with negligible emittance. The coherent angular spectral energy radiated from the bunch is given by

\[
\frac{d^2U_c}{d\mathbf{k}d\Omega} = N_b^2 \chi(\theta) F(k) \frac{d^2U}{d\mathbf{k}d\Omega},
\]

(2)

where \(\chi(\theta)=1\) is the divergence factor\(^\text{14}\) set to unity for a cold beam, \(F(k)\) is the e-beam “form factor,” calculated by taking the spatial Fourier transform of the charge distribution of the e-beam \(f(x)\) normalized to unity \(\int f(x)dx = 1\). The complex-valued square-root of the form factor, sometimes called the “structure factor” is written as

\[
\sqrt{F(k)} = \int f(x)e^{-ik\mathbf{x}}dx,
\]

(3)

where \(F(k) = |\sqrt{F(k)}|^2\). With Eq. (3), the CTR field distribution can be calculated for an arbitrary e-beam distribution. In a linear plasma fluid model we write a general, continuous distribution of the form

\[
f(x) = f_\perp(z)f_\perp(z) + \text{Re}\{f_\parallel(x)\},
\]

(4)

where \(f_\perp(z)\) and \(f_\perp(z)\) are the unmodulated longitudinal and transverse distributions, respectively. Microbunching and higher-order beam modulations are described by the perturbation \(f_\parallel(x)\), which is small such that \(\text{Re}\{f_\parallel\} \ll f_0\). For simple bunching geometries, such as those that typically occur in high-gain FELs lasing at the fundamental mode, \(f_\parallel(x)\) can be expressed as a simple one-dimensional sinusoidal longitudinal modulation.\(^\text{2,10}\) For more exotic higher-order bunching structures however, a more elaborate description is needed, and it is advantageous to represent the modulation as a superposition of 3D spatial modes. Since microbunching along \(z\) necessarily occurs for the types of beam interactions we focus on here (on top of any transverse rearrangement of the distribution) the analytic bunching function is expanded into a complete basis set of transverse modes together with a longitudinal modulation as follows:

\[
f_\parallel(x) = \sum_{k_p} \sum_{l} c_{l}^{(b)} \Phi_{l}^{(b)}(x) e^{ik_p z}.
\]

(5)

The harmonics \(h\) of the fundamental spatial microbunching frequency \(k_p = 2\pi/\lambda_0\) are included in the expansion sum over the harmonic mode amplitudes \(c_{l}^{(b)}\) with the basis functions \(\Phi_{l}^{(b)}(x)\). The mode indices are \(p\) and \(l\). The characteristic modulation wavelengths are assumed small compared to the unmodulated dimensions of beam. Modal expansions of this sort have been employed previously in the context of e-beam modulators\(^\text{7}\) and FELs\(^\text{15,16}\) to describe the e-beam modulation due to the interaction with the resonant signal field. Here we choose a basis set of Laguerre–Gaussian (LG) functions, which also arise in the solutions to paraxial electromagnetic waves, and thus provide a convenient working basis, particularly for e-beams that interact with free-space EM waves with pulse lengths much longer than the e-beam. In cylindrical coordinates \((\rho, \varphi, z)\) the transverse basis functions are composed of radial modes \(p\) and azimuthal modes \(l\) and are written as

\[
\Phi_{l}^{(b)}(x) = \frac{e^{-\rho^2/2\sigma_p^2}}{2\pi\sigma_p^2} e^{\rho^2/2\sigma_p^2} x^l e^{i\rho^2/2\sigma_p^2} \sin^2 \theta \exp[-k^2(\sigma_r^2 \cos^2 \theta + \sigma_a^2 \sin^2 \theta)],
\]

(6)

where \(\sqrt{\sigma_a}\) is the root-mean-squared (rms) radius of the fundamental mode \((p, l=0)\) module. The functions are orthogonal via \(\int \Phi_{p}^{(b)}(r) \Phi_{p'}^{(b)}(r') dx = \delta_{pp'} \delta_{ll'} (p+|l|)!/4\pi \sigma_p^2 \sigma_p^2\). The phase of \(\exp(il\varphi)\) together with the harmonic longitudinal modulation describes helical microbunching. Positive values of \(l\) describe a helical modulation that coils in a left-handed sense moving in the positive direction along the axis \(z\).

We approximate the unmodulated bunch distributions as normalized Gaussians and write

\[
f_\perp(z) = \frac{1}{\sqrt{2\pi\sigma_z^2}} e^{-z^2/2\sigma_z^2},
\]

(7)

With the distributions in Eqs. (6) and (7), the normalization requirement \(\int f(x)dx = 1\) is strictly satisfied, but is well-approximated if \(\sum_{p,l} c_{l}^{(b)} \exp[-(h_k \sigma_1)^2]/2 < 1\). Accordingly, for the remainder of this paper it is assumed that \(k_0 \sigma_{\|} \gg 1\) such that this condition holds. Note that, if only the fundamental beam mode \((p, l=0)\) is present, then \(\Phi_{0}^{(b)}(x) = f_\perp(z)\) and Eq. (4) can be written in the simplified form of Ref. 10

\[
f(x) = e^{-\rho^2/2\sigma_p^2} e^{-z^2/2\sigma_z^2} \left[1 + \text{Re}\left(\sum_{l=1}^{\infty} c_{l}^{(b)} e^{il\varphi} \right)\right],
\]

(8)

and the coefficients \(c_{l}^{(b)}\) play a role analogous to that of the longitudinal bunching factors at each harmonic in standard microbunching scenarios. The results from Ref. 10 follow directly from this reduced form.

The structure factor \(\sqrt{F(k)}\) in Eq. (3) is evaluated analytically with the distributions in Eqs. (6) and (7) with the exponential phase given by \(n \cdot \mathbf{x} = \rho \cos \varphi \sin \theta \cos \theta + \rho \sin \varphi \sin \theta \sin \phi + \rho z \cos \theta\). For emission at wavelengths comparable to the finite bunch length \((k_0 \sigma_{\|} \sim 1)\), the structure factor, and therefore the CTR spectrum, is dominated by radiation from the unmodulated component of the e-beam in Eq. (7). The integral in Eq. (3) is evaluated to obtain the CTR spectrum for the entire bunch

\[
\frac{d^2U_c}{d\mathbf{k}d\Omega} = \frac{N_b^2 \chi(\theta) 2 \sin^2 \theta \exp[-k^2(\sigma_r^2 \cos^2 \theta + \sigma_a^2 \sin^2 \theta)]}{4\pi^2 \epsilon_0} \left(1 - \beta^2 \cos^2 \theta\right)^2.
\]

(9)

The characteristic hollow intensity distribution in the forward direction is suppressed for wide opening angles \(\theta < 1/\gamma\) due to finite beam size effects. Note, however, that when \(\sigma_r = \sigma_a\), the finite e-beam distribution is symmetric and has no effect on the angular distribution, which becomes precisely that of single particle emission. Otherwise, when \(\sigma_p \gg \sigma_r, \sigma_a\), and \(\sigma_p = \sigma_r, \sigma_a\), within the narrow forward cone, the spectral emission is determined to first order by \(dU_c/dk = (N_b^2 e^2/4\pi^2 \epsilon_0) (\sigma_r^2 - \sigma_a^2/2k^4) \exp(-k^2 \sigma^2_r)\). It should be noted that the finite beam distribution can have an effect on the
spectral emission since, for larger beams, the high frequency coherent emission is pushed nearer to the axis. Combined with the axial null of the single particle TR distribution, this effect tends to suppress the higher frequencies and thus may affect certain diagnostics aimed at measuring the overall bunch length from the width of the raw CTR emission spectrum.

At emission wavelengths on the order of the microbunching scales \((k \sim h k_i)\), the modulated component of the beam is obviously the dominant contribution to the coherent radiation. The structure factor is calculated with Eqs. (5) and (6) in terms of the modal density expansion amplitudes and is

\[
\sqrt{F(k)} = \frac{1}{2} \sum_{h=1}^{\infty} \sum_{l=0}^{\infty} c_{\lambda,l}^{(b)}(k \sigma_p \sin \theta)^{|l|} L_p^{(0)}((k \sigma_p \sin \theta)^2) \\
\times e^{-\frac{1}{2} (h k_0 - k \cos \theta)^2} - \frac{\sigma^2}{2} (k \sin \theta)^2 + 2 \pi i (\phi - \pi / 2). \tag{10}
\]

Inserting this into Eq. (2) we obtain a general expression for the coherent emission from the modulated beam

\[
\frac{\partial^2 U_C}{\partial k \partial \Omega} = \frac{\pi^2 e^2}{16 \pi^2 \epsilon_0} \sin^2 \theta \left(1 - \beta^2 \cos^2 \theta \right)^2 \\
\times \sum_{l=1}^{\infty} \sum_{h=1}^{\infty} c_{\lambda,l}^{(b)} e^{i l \phi} (k \sigma_p \sin \theta)^{|l|} \\
\times L_p^{(0)}((k \sigma_p \sin \theta)^2) \exp \left[-\frac{\sigma^2}{2} (h k_0 - k \cos \theta)^2 \right] \tag{11}
\]

Several important features are noted about the structure factor in this form. The first is that the emission is strongly peaked near \(k \sim h k_i\) for small forward angles, as expected. Second, and central to the theme of this work, is the azimuthal phase dependence of the field emission for beams with azimuthal structure \(l \neq 0\). These terms have an azimuthal phase dependence \(\exp(i l \phi)\) in the field that corresponds directly to the azimuthal mode number of the e-beam (Fig. 1). An e-beam with a single twist helical bunching modulation \(l=1\), for example, will radiate a field with a wavefront that also has an \(l=1\) helically evolving phase. Thus, similar to the case of the FEL, the CTR emission from a helically microbunched beam also contains nonzero optical OAM about the axis of propagation, though for CTR the distribution is modified by additional factors. The intensity distribution is weighted by the angular factor of \((\sin \theta)^{|l|+2}\) for the fundamental radial mode \(p=0\) (and additional factors of \(\sin \theta\) for higher radial modes from the Laguerre polynomials), which pushes the emission peak outward into larger opening angles for higher-order \(l\) modes, similar to the case of higher-order modes in paraxial optics. This is discussed in more detail later.

![FIG. 1.](color online) Beam density distributions displaying the structure of helical microbunching for (a) \(l=0\), (b) \(l=1\), and, (c) \(l=2\) modes. The bunching factor in each case is \(b_l^{(j)}=40\%\). The isosurface encloses 90% of the electrons.

### 1. Bunching factor

To elucidate the effect of azimuthal e-beam microbunching structures on the CTR distribution, it is useful to re-express the mode expansion amplitudes in Eq. (11) in terms of a bunching factor for each azimuthal mode. The azimuthal bunching factor \(b_l^{(j)}\) describes the normalized amplitude of the density modulation at the discrete mode \(l\) via

\[
b_l^{(j)} = \int f_l(x) e^{-i l \phi} d^2 x. \tag{12}
\]

The frequency content is found from the transformed bunching factor \(\tilde{b}_l^{(j)} = \int f_l(x) e^{-i \phi} dz\). Since the density distribution
is given by discrete harmonics of the fundamental frequency $k_b$, we write
\[ b_l(z) = \sum_{h=1}^\infty b_l^{(h)} e^{ihk_bz}, \] (13)
where the azimuthal bunching factor at a given harmonic is therefore written as
\[ b_l^{(h)} = \frac{k_b}{2\pi} \int_0^{2\pi k_b} b_l(z) e^{-ihk_bz} dz = \sum_{p=0}^\infty c_{p,l}^{(h)} \Phi_p(x) e^{-i\theta d} d\theta. \] (14)

With Eq. (5) we obtain an explicit expression for the bunching factor in terms of the harmonic mode amplitudes $c_{p,l}$:
\[ b_l^{(h)} = \sum_{p=0}^\infty c_{p,l}^{(h)} \left( \frac{||l||}{l} \right)^2 \frac{2^{l+1}}{l+1} \right), \] (15)

where $2F_1(a;b;c;x) = \sum_{n=0}^\infty (a)_n (b)_n x^n / (c)_n n!$ is the hypergeometric series and $(a)_n = a(a+1)(a+2)\ldots(a+n-1)$ is the rising factorial. In the event that the density modulation is described by only the fundamental radial mode, Eq. (15) becomes
\[ b_l^{(h)} = \left( \frac{||l||}{l} \right)^2 \frac{2^{l+1}}{l+1} c_{l,l}^{(h)} \] (p = 0) (16)

Note that if only the fundamental expansion mode $p,l=0$ is nonzero then $b_0^{(h)} = c_0^{(h)}$.

A direct analytic representation of the angular spectrum in terms of $b_l^{(h)}$ is obtained by assuming that the fundamental radial mode ($p=0$) is the dominant mode in the beam and ignoring the others. This simplification allows direct examination of the coherent emission for pure azimuthal structures. If, of course, the individual mode expansion coefficients $c_{p,l}^{(h)}$ are known, one can easily calculate both $b_l^{(h)}$ and $\gamma(k)$ generally. In terms of the reduced bunching factor in Eq. (16) the angular spectrum for the fundamental radial beam mode is
\[ \frac{d^2U_C}{dkd\Omega} = \frac{N^2 e^2}{16 \pi^2 e_0 \theta^2} \left( \frac{\sin^2 \theta}{1 - \beta^2 \cos^2 \theta} \right) \times \sum_{h=1}^\infty \sum_{l=0}^\infty b_l^{(h)} e^{i(l+2m)} \left( \frac{k \sigma_x \sin \theta}{\sqrt{2}} \right)^{||l||} \times \exp \left[ -\frac{\sigma^2}{2} (hk_b - k \cos \theta) \right] \left[ -\frac{\sigma^2}{2} (k \sin \theta) \right]^{2}. \] (17)

Note that the angular spectral energy distribution may have an azimuthal dependence only if the microbunching is described by more than one azimuthal mode since the only $\phi$ dependence in the spectral field is in the phase.

**2. Angular distribution**

The spectral emission profile is peaked at an angle $\theta_m$, which is a function of the e-beam size and frequency and is modified by the presence of the higher-order azimuthal modes in the beam. For a single azimuthal mode and harmonic, it is given approximately by
\[ \theta_m \approx \sqrt{\frac{||l|| + 1}{2 \gamma^2 + \sigma_r^2 k^2 - \sigma_z^2 k^2 (1 - h k_b/k)}} \] (18)

which, for emission at the bunching frequency harmonic, is independent of the bunch length $\sigma_z$. At this frequency the angular spread is determined by the transverse beam size, provided $\sigma_r k_b$ is not negligible compared to $\gamma$. In fact when $\sigma_r k_b/\gamma^2 > 1$, it is clear that increased transverse beam sizes result in narrowing of the forward angular distribution. This can be understood by considering the relative phase of the individual electrons emitting across the transverse bunch face. The coherent emission angle is forced closer to the axis for larger beams because of the more pronounced phase difference interference generated by large transverse offsets. It is also interesting to note the dependence of $\theta_m$ on the mode $l$, where it is clear that higher-order helical bunching modes emit into larger opening angles (Fig. 2). Accordingly, Eq. (18) identifies the angular peaks for beams that satisfy the small angle approximation for $\theta$, but beams with arbitrarily large values of $l$ may violate this assumption, and the angular peak should be found from the full expression in Eq. (17).

**3. Frequency distribution**

Much like the angular distribution, the azimuthal modes in the e-beam also modify the frequency distribution of the CTR emission, though to a lesser extent. Naturally, the emission from a finite bunch has a sharp spectral peak near the microbunching frequency, but weighted by the forward emission angle and profile distribution. For any arbitrary small angle $\theta$, the maximum frequency peak is given by
\[ k_m \approx \frac{h k_b}{1 + (\sigma_r \gamma / \sigma_z)^2 + \frac{||l||}{(\sigma_r h k_b)^2}} \] (19)

where it has again been assumed that $\sigma_r k_b/\gamma > 1$. It is clear that the value of the $l$ bunching mode has little effect on the
frequency distribution, save for very large \( l \) but in which case the small angle assumption may no longer hold. In beams where \(|l| \ll (\hbar k_b \sigma_z)^2\) the frequency peak is roughly

\[
\frac{k_m}{\hbar k_b} \approx 1 - (\sigma_p \theta / \sigma_z)^2, \tag{20}
\]

when \((\sigma_p \theta / \sigma_z)^2 \ll 1\). As expected, the CTR emission spectrum is redshifted quadratically in angle off-axis.

It is interesting to note the dependence of \(k_m\) on the transverse beam size \(\sigma_p\), which shows that larger transverse beam profiles correspond to greater redshifting of the emission for a fixed bunch length and observation angle. Analogous to the narrowing of the forward angular emission peak, frequencies on the high end of the microbunching emission band are suppressed by increased beam profiles and the overall emission spectrum tends toward longer wavelengths. The result is redshifting of the CTR microbunching spectrum compared to the coherent FEL spectrum (for which the frequency maximum is on-axis for the fundamental eigenmode of the system), even though the radiation is emitted from the same microbunched beam. This may become important in measurements where CTR emission is used as a microbunching diagnostic. This type of discrepancy can be avoided in general if \(\sigma_p / \gamma \sigma_z \ll 1\).

4. Total energy

The integral of Eq. (17) over frequency is simplified in the limit where the emission spectrum approaches a delta function, \(\delta(k-\hbar k_b)\). The frequency-integrated angular distribution from e-beams in which a single azimuthal microbunching mode is excited is shown in Fig. 2. The angular peak in the emission is pushed into larger opening angles with higher-order \( l \)-modes. Also apparent is the notable rise in total energy emission for these modes. This behavior is borne out analytically by calculating the total CTR energy for each structure, where angular integral over the narrow forward distribution is simplified by assuming \(\sigma_p \theta / \sigma_z < \sigma_p / \gamma \sigma_z \ll 1\) across the radiation distribution. The result is an expression for the total emission energy for a helically microbunched beam at a single mode \( l \) and harmonic \( h \)

\[
U_{C,l}^{(b)} = \frac{N_s e^2}{16 \pi \epsilon_0 \sigma_z} \left| \langle \rho \rangle \right|^2 \left( \frac{\gamma}{\sigma_p \hbar k_b} \right)^4 \frac{(|l|+1)!}{(|l|/2)!!^2} \right. \right. \tag{21}
\]

At the fundamental spatial mode \( l=0 \), the expression for the emission energy of a purely longitudinally microbunched beam is obtained, including the characteristic \((\gamma / \sigma_p \hbar k_b)^4\) scaling.\(^{10}\) This scaling applies in the limit \(\gamma / \sigma_p \hbar k_b \ll 1\), where the microbunching wavelength in the beam frame is much smaller than the transverse beam size. Otherwise, for higher-energy beams (or equivalently for beams that are small transversely, as described below), the scaling is somewhat weaker than \((\gamma / \sigma_p \hbar k_b)^4\) and is found through the full solutions of Eq. (17). For higher-order azimuthal modes that satisfy Eq. (21), the total energy in the forward direction is actually greater than for lower order modes since the factor \((|l|+1)! / (|l|!/2)!!^2\) grows with \(|l|\). This effect is the result of the increased overlap of the form factor of these beam modes with the angular distribution kernel of the single particle TR. Physically, the higher-order modes naturally emit into larger opening angles [Eq. (18)], delivering a larger fraction of the total power into the region outside the intrinsic axial null of the TR where the emission is suppressed.

Equation (21) is a useful approximation for the total emission energy, in the relevant limits. However, since the higher modes emit into the larger emission angles, the small angle assumption used in obtaining Eq. (21) may not be strongly satisfied, particularly for reduced transverse beam sizes, which also emit into larger angles. Figure 3 illustrates this effect, and shows the scaling of the total energy with \(\sigma_p\) for three different azimuthal modes. The \(1/\sigma_p^4\) behavior is evident, both in the full numerical solutions obtained from Eq. (17) and in the approximate analytic solution in Eq. (21). At large \(\sigma_p\) (specifically large \(\sigma_p \hbar k_b / \gamma\)) the agreement is good, particularly for lower \( l \)-mode beams. But when beams emit into larger angles the agreement dissolves. In fact, the small \(\theta\) assumption begins to fail dramatically where the complete solution actually predicts a lower total emission energy for increasing \( l \), while the approximate solution in Eq. (21) erroneously predicts the opposite. Mathematically the disagreement simply shows where the assumptions made in deriving the simple scaling begin to break down. Physically the drop in emission energy can be understood from the perspective of the virtual photon model of emission.\(^{17}\) It is attributed, in part, to the increase in the helical pitch angle of the helical density distribution. As the beam radius shrinks, the helical winding of the density distribution becomes steeper. This changes the direction of the space-charge fields and eventually reduces the effective fringing fields that drive the CTR emission. The pitch angle of the helical winding is greater for the higher \( l \) modes, so the total emission energy is reduced for them first as the radius shrinks. In the limit where the pitch angle approaches the \(\pi/2\) maximum (a zero pitch angle describes longitudinal bunching at \( l=0\)), the beam is effectively debunched and the coherent emission energy at \(\hbar k_b\) drops toward zero. There is thus a balance be-
between the increase in emission energy for the higher-order modes which emit off-axis, and the decrease in energy that results from the modified fringing fields. This is a subject for future investigations.

B. Conclusion

The spectral and angular distribution of CTR emitted from an e-beam with higher-order spatial microbunching geometries has been derived using a form factor description in a cold fluid model. Helical bunching geometries are shown to carry an additional azimuthal phase in the radiation field that corresponds to the azimuthal modes excited in the beam. The possibility of generating and utilizing the optical OAM from such sources is suggested, particularly where limitations preclude generation of these modes by other means. Such beams also emit into larger forward angles, similar to the case of higher-order paraxial laser modes, and may also radiate at a shifted frequency from the fundamental bunching frequency, depending on the unmodulated beam distribution parameters. These emission features, perhaps in conjunction with the phase geometry, may be used to distinguish and identify higher-order microbunching structures in the e-beam. To identify the presence of a single dominant helical microbunching mode in the beam, for example, one might use an interferometric mode sorter similar to that described in Ref. 18.

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