Overview of Beam Physics

Lecture 2
Electrodynamics

• Start with Maxwell equations (mks)
  \[ \nabla \cdot \vec{B} = 0, \quad \nabla \cdot \vec{D} = \rho_e, \quad \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}_e, \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]

  Where the fields obey the constitutive relations
  \[ \vec{D} = \varepsilon(\vec{D})\vec{E}, \quad \vec{B} = \mu(\vec{H})\vec{H} \]

• Continuity of sources implied \( \nabla \cdot J_e + \frac{\partial \rho_e}{\partial t} = 0 \)

• Charged particles obey the Lorentz force equation,
  \[ \frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B}) \]

  With a generalization of the momentum \( p \).
Relativistic dynamical quantities

- Velocity normalized to speed of light \( \vec{\nu} \equiv \beta c \)
- Momentum is defined relativistically as
  \[
  \vec{p} = \gamma m_0 \vec{\nu} \equiv \beta \gamma m_0 c
  \]
  with \( m_0 \) being the rest mass (nonrel. limit), and
  \[
  \gamma \equiv \left( 1 - \left( \frac{\vec{\nu}^2}{c^2} \right) \right)^{-1/2}
  \]
- The energy \( U \) and momentum are related by 4-vector invariant length (like space-time)
  \[
  \vec{p}^2 c^2 - U^2 = -\left( m_0 c^2 \right)^2 \quad U = \gamma m_0 c^2
  \]
Relativistic equations of motion

• Purely magnetic forces yield no change in energy, as force is normal to instantaneous motion, \( dU = \vec{v} \cdot d\vec{p} \)

• Transverse equation of motion has larger inertial mass
  \[
  \gamma m_0 \frac{d\vec{v}}{dt} = q(\vec{v} \times \vec{B})
  \]

• Longitudinal acceleration with electric field gives change in \( \gamma \), different scaling of inertial mass
  \[
  \gamma^3 m_0 \frac{dv_\parallel}{dt} = qE_0
  \]
EM Fields in special relativity

- Longitudinal fields are “Lorentz invariant”
- Transverse fields (i.e. self fields in beam)

\[ \vec{E}_\perp = \gamma \left( \vec{E}_\perp' - \vec{v} \times \vec{B}_\perp' \right) \]
\[ \vec{B}_\perp = \gamma \left( \vec{B}_\perp' + \frac{1}{c^2} \vec{v} \times \vec{E}_\perp' \right) \]

- In beam rest frame \( \vec{B}_\perp' = 0 \) and
\[ \vec{F}_\perp = q \left( \vec{E}_\perp + \vec{v} \times \vec{B}_\perp \right) = q\vec{E}_\perp (1 - \beta^2) = \frac{q\vec{E}_\perp}{\gamma^2} \]

“extra” factors of inertial mass…
Hamiltonians in special relativity

- A Hamiltonian is a function of coordinates and “canonical momentum” which gives the equations of motion
  \[ \dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}. \]

- It is constructed from a Lagrangian
  \[ H(\vec{x}, \vec{p}) = \vec{p} \cdot \vec{x} - L \]
  \[ p_i \equiv \frac{\partial L}{\partial \dot{x}_i} \]

- A Hamiltonian can rigorously give equations of motion, and constants of the motion, i.e. \( H \),
  \[ \dot{H} = \frac{\partial H}{\partial t}; \quad H \text{ independent of time is constant.} \]
Relativistic dynamics

• Need potentials \( (\phi_e, \vec{A}) \) for variational analysis

\[
\vec{E} = -\vec{\nabla}\phi_e - \frac{\partial\vec{A}}{\partial t} \quad \vec{B} = \vec{\nabla} \times \vec{A}
\]

• The Lorentz force can be written as

\[
\vec{F}_L = \frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B})
\]

\[
= q\left[-\vec{\nabla}\phi_e - \frac{\partial\vec{A}}{\partial t} - (\vec{v} \cdot \vec{\nabla})\vec{A}\right] = q\left[-\vec{\nabla}\phi_e - \frac{d\vec{A}}{dt}\right]
\]

• Relations between energy/momenta and potentials can be seen

\[
p_{c,i} = p_i + qA_i \quad H = U + q\phi_e
\]
The relativistic Hamiltonian

- From the 4-vector relation of energy and momentum,

\[ (H - q\phi_e)^2 = (\vec{p}_c - q\vec{A})^2 c^2 + (m_0 c^2)^2 \]

or

\[ H = \sqrt{(\vec{p}_c - q\vec{A})^2 c^2 + (m_0 c^2)^2 + q\phi_e} \]

- Canonical variables simply related to familiar mechanical variables \((U,p)\)
Transformations of Hamiltonian

- Transform to new canonical variables to make the Hamiltonian constant?
- Example: wave with phase velocity $v_\phi$
- Galilean transformation $\zeta = z - v_\phi t \quad p_\zeta = p_z$
- New Hamiltonian $\tilde{H}(\zeta, p_\zeta) = H(\zeta, p_\zeta) - v_\phi p_\zeta$.
- Example: traveling wave in accelerator…
Traveling wave Hamiltonian

- Field and vector potential
  \[ E_z(z - v_\phi t) = -\frac{\partial A_z}{\partial t} = E_0 \sin[k_z(z - v_\phi t)], \quad A_z(z - v_\phi t) = -\frac{E_0}{k_z v_\phi} \cos[k_z(z - v_\phi t)] \]

- 1D COM Hamiltonian:
  \[ H = \sqrt{\left(p_{z,c} + \frac{qE_0}{k_z v_\phi} \cos[k_z(z - v_\phi t)]\right)^2 c^2 + (m_0 c^2)^2} \]

- In terms of mechanical momentum for plotting (“algebraic” Hamiltonian, not for equations of motion)
  \[ \tilde{H}(\xi, p_\xi) = \sqrt{p_\xi^2 c^2 + (m_0 c^2)^2} - v_\phi p_\xi + \frac{qE_0}{k_z} \cos[k_\xi] \]
Phase plane plots

- With algebraic form, for a given $H$, choose $\xi$, can determine $p$.
- Plot motion along constant $H$ contours
- Example: longitudinal motion in small “potential” case, typical of ion linacs and circular accelerators
Phase space, general considerations

- Phase space density distribution: \( f(\bar{x}, \bar{p}, t) \)
- Number of particles near a phase space point is \( f(\bar{x}, \bar{p}, t) d^3x d^3p \)
- In beam physics, we often will have 2D projections of phase space, i.e. \( (x, p_x) \)
- Motion uncoupled in \( x, y, \) and \( z \).
Liouville theorem

• Vlasov equation
  \[ \frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{x} \cdot \nabla_x f + \dot{p} \cdot \nabla_p f \]

• For Hamiltonian systems:
  \[ \frac{df}{dt} = \frac{\partial f}{\partial t} + \sum \left( \frac{dx_i}{dt} \frac{\partial f}{\partial x_i} + \frac{dp_i}{dt} \frac{\partial f}{\partial p_i} \right) \]
  \[ = \frac{\partial f}{\partial t} + \sum \left( \frac{\partial H}{\partial p_i} \frac{\partial f}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial f}{\partial p_i} \right) \]
  \[ = \frac{\partial f}{\partial t} + \sum \left( \frac{\partial H}{\partial p_i} \frac{\partial f}{\partial x_i} \frac{dH}{dx_i} - \frac{\partial H}{\partial x_i} \frac{\partial f}{\partial p_i} \frac{dH}{dp_i} \right) = \frac{\partial f}{\partial t} \]

• If particle number is conserved \( \frac{\partial f}{\partial t} = 0 \)

• Conservation of phase space density: \( \frac{df}{dt} = 0 \).
Design trajectory and paraxial rays

- Trajectories measured with respect to idealized design
- Small angles assumed; paraxial rays
  \[ p_{x,y} \ll p_z \equiv |\vec{p}| \]
- Equations of motion parameterized using \( z \) as independent variable instead of \( t \):
  \[
  x' = \frac{dx}{dz} = \frac{1}{v_z} \frac{dx}{dt} = \frac{x}{v_z}
  \]
Example: equations of motion in magnetic quadrupole

- Magnetic field (normal)
  \[ \vec{B}_2 = B'(y\hat{x} + x\hat{y}) \]

- Magnetic force
  \[ \vec{F}_\perp = qv_z \vec{z} \times \vec{B}_2 = qv_z B'(y\hat{y} - x\hat{x}) \]

- Equation of motion
  \[
  x'' = \frac{F_x}{\gamma m_0 v_0^2} = -\frac{qB'}{p_0} x \quad
  y'' = \frac{F_y}{\gamma m_0 v_0^2} = \frac{qB'}{p_0} y
  \]

- Simple harmonic oscillator form:
  \[ x'' + \kappa_0^2 x = 0 \quad \kappa_0^2 \equiv \frac{qB'}{p_0} = B'/BR \]

  Solution:
  \[ x = x_0 \cos[\kappa_0 (z - z_0)] + \frac{x_0'}{\kappa_0} \sin[\kappa_0 (z - z_0)] \]
Motion in uniform magnetic field

- Field $\vec{B} = B_0 \hat{z}$
- Lorentz force
  $$\frac{dp_\perp}{dt} = 0, \quad \frac{dp_z}{dt} = q(\vec{v}_\perp \times \vec{B}) = \frac{qB_0}{\gamma m_0} (\vec{p}_\perp \times \hat{z})$$
- Uniform motion in $z$
- Transverse circle at cyclotron frequency.
  $$\frac{d^2v_x}{dt^2} + \omega_c^2 v_x = 0, \quad \frac{d^2v_y}{dt^2} + \omega_c^2 v_y = 0. \quad \omega_c = \frac{qB_0}{\gamma m_0}$$
- Radius related to $p_\perp$
  $$p_\perp (\text{MeV}/c) = 299.8 \cdot B_0 (\text{T}) R (\text{m})$$

Define: $BR \equiv p_\perp (\text{MeV}/c) / 299.8$

- Helix in general, angle $p_z / p_\perp$
Solenoid focusing

- Particles pick up $p_\perp$ in fringe field

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho B_\rho = -\frac{\partial B_z}{\partial z} \quad \text{or} \quad B_\rho \approx -\frac{\rho}{2} \frac{\partial B_z}{\partial z} \bigg|_{\rho=0}$$

- Integrate, to find

$$\Delta p_\varphi \approx q \int_{z_i}^{z_f} v_z B_\rho \, dt = q \int_{z_i}^{z_f} B_\rho \, dz = -q \frac{\rho_0}{2} \int_{z_i}^{z_f} \frac{\partial B_z}{\partial z} \bigg|_{\rho=0} \, dz = -q \frac{\rho_0}{2} \int_{z_i}^{z_f} dB_z \bigg|_{\rho=0} \, dz$$

$$= -q \frac{\rho_0}{2} [B_z(z_f) - B_z(z_i)] = -q \frac{\rho_0}{2} B_0$$

- Result known as Busch’s theorem

- Assume particles are “born” in region with no magnetic field…
The Larmor Frame

- Note radius of curvature $R$ is 1/2 of offset $\rho$. Particle passes through origin!
- A frame defined by particle and origin rotates with Larmor frequency
  \[ \omega_L \equiv \frac{d\theta_L}{dt} = \frac{\omega_c}{2} = \frac{qB_0}{2\gamma m_0} \]
- In Larmor frame, equations of motion are simple harmonic
  \[
  \begin{align*}
  \ddot{x}_L + \omega_L^2 x_L &= 0, \\
  \ddot{y}_L + \omega_L^2 y_L &= 0,
  \end{align*}
  \]
  \[
  \begin{align*}
  x''_L + k_L^2 x_L &= 0, \\
  y''_L + k_L^2 y_L &= 0,
  \end{align*}
  \]
Distributions and equilibrium

- A distribution can be in equilibrium under application of *linear* force
- Separability:  
  \[ f(\vec{x},\vec{p}) = N f_x(x,p_x) f_y(y,p_y) f_z(z,p_z) \]
- Vlasov equilibrium  
  \[ \ddot{X} \frac{dX}{dx} P + F_x \frac{dP}{dp_x} = 0 \]
- Separated equations:  
  \[ \frac{1}{X F_x(x)} \frac{dX}{dx} = -\frac{\gamma_0 m_0}{pP} \frac{dP}{dp_x} = \lambda_x \]
- Momentum solution  
  \[ P(p_x) = C_p \exp\left(-\frac{p_x^2}{2\gamma_0 m_0 k_B T_x}\right) \equiv C_p \exp\left(-\frac{p_x^2}{2\sigma_{p_x}^2}\right) \]
- Solution for linear restoring force in \( x \):  
  \[ x(x) = C_x \exp\left(-\frac{\gamma_0 m_0 v_0^2 k_B^2 x^2}{2 k_B T_x}\right) \equiv C_x \exp\left(-\frac{x^2}{2\sigma_x^2}\right) \]
- Self-forces (space-charge) are nonlinear in \( x \), *Maxwell-Vlasov* equilibria not Gaussian.
RMS envelope equation

- Want information on extent of distributions
- Look at RMS envelope

\[ \sigma_x^2 = \langle x^2 \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 f_x(x,x') \, dx \, dx', \]

- Take derivatives

\[ \frac{d\sigma_x}{dz} = \frac{d}{dz} \sqrt{\langle x^2 \rangle} = \frac{1}{2\sigma_x} \frac{d}{dz} \langle x^2 \rangle \]

\[ = \frac{1}{2\sigma_x} \frac{d}{dz} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 f_x(x,x') \, dx \, dx' \]

\[ = \frac{1}{\sigma_x} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xx' f_x(x,x') \, dx \, dx' = \sigma_{xx}'. \]

\[ \frac{d^2\sigma_x}{dz^2} = \frac{d}{dz} \frac{d\sigma_x}{dz} = \frac{1}{\sigma_x} \frac{d}{dz} \frac{d\sigma_x}{dz} - \frac{\sigma_{xx}^2}{\sigma_x^3} \]

\[ = \frac{1}{\sigma_x} \frac{d}{dz} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xx' f_x(x,x') \, dx \, dx' - \frac{\sigma_{xx}^2}{\sigma_x^3} \]

\[ = \sigma_x^3 \langle xx'' \rangle - \frac{\sigma_{xx}^2}{\sigma_x^3}. \]
RMS Emittance and the Envelope

• The RMS envelope equation can be put in standard form by noting that the *rms emittance*

\[ \varepsilon_{x,\text{rms}}^2 = \langle x^2 \rangle \langle x'^2 \rangle - \langle xx'' \rangle^2 \]

is constant under of linear forces \( x'' + \kappa_x^2 x = 0 \)

• Also, with this assumption, \( \langle xx'' \rangle = -\kappa_x^2 \langle x^2 \rangle = -\kappa_x^2 \sigma_x^2 \)

and

\[ \sigma_x'' + \kappa_x^2 \sigma_x = \frac{\varepsilon_{x,\text{rms}}^2}{\sigma_x^3} . \]

• The emittance enters the envelope equation as a pressure-like term

• Example solution: equilibrium \( \sigma_{eq} = \sqrt{\frac{\varepsilon_{x,\text{rms}}}{\kappa_x}} \)
Inclusion of Acceleration

- In electron sources we have strong acceleration
- Acceleration introduces “adiabatic” damping
  \[
  \frac{d^2 x}{dz^2} = \frac{d}{dz} \frac{p_x}{p_z} = -\kappa_x^2 x' + \frac{(\beta \gamma)}{\beta \gamma} x' \,, \quad (\beta \gamma)' = F_z = \frac{F_z}{p}
  \]
- This enters into the envelope as
  \[
  \frac{d^2 \sigma_x}{dz^2} = \frac{d}{dz} \frac{\sigma_{xx'}}{\sigma_x} = \frac{1}{\sigma_x} \frac{d \sigma_{xx'}}{dz} - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \frac{1}{\sigma_x} \left[ \sigma_{xx'}^2 - \kappa_x^2 \sigma_x^2 - \frac{(\beta_0 \gamma_0)'}{\beta_0 \gamma_0} \right] - \frac{\sigma_{xx'}^2}{\sigma_x^3}
  \]
- In standard form
  \[
  \frac{d^2 \sigma_x}{dz^2} + \frac{(\beta \gamma)'}{\beta \gamma} \frac{d \sigma_x}{dz} + \kappa_x^2 \sigma_x = \frac{\varepsilon_{n,x}^2}{(\beta \gamma)^2 \sigma_x^3}
  \]
  where we have introduced the normalized emittance
  \[
  \varepsilon_{n,x} = \beta \gamma \varepsilon_{x,rms}
  \]
Busch’s theorem: magnetization emittance

- If particles are “born” in magnetic field, then they have canonical angular momentum. Upon leaving field, they have rms transverse momentum $\sigma_{p\perp} \approx \frac{qB_0}{2} \sigma_{x(y)}$
- This can be translated to normalized emittance

$$\varepsilon_{n,x} \approx \frac{\sigma_{p\perp}}{m_0c} \sigma_x \approx \frac{qB_0}{2m_0c} \sigma_x^2 \approx 150B_0(T)\sigma_x^2$$
Reading references

• Review Chapter 1 sections (all) concerning dynamics and phase space
• Review Chapter 2 sections (1, 3-6) concerning motion in magnetic fields and linear focusing
• Review Chapter 5 sections (1-3, 5) concerning distributions and envelopes