Analysis of Halbach Segmented Pure Permanent Magnet Quadrupole

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The ideal Halbach quad

- Imagine a permanent magnet geometry in which the magnetization varies continuously with $\phi$, as shown

$$\vec{M} = M_0 (\hat{\rho} \sin(2\phi) + \hat{\phi} \cos(2\phi))$$

- This model problem can be used to generate the segmented-piece solutions

- In order to analyze this geometry, it is best to introduce Amperian currents...
Amperian currents:

- The equivalent currents are found in a sheet on the inner and outer radius ($r_1$ and $r_2$)

\[ \vec{K} = \vec{M} \times \hat{n} \]

\[ K_z = \pm M_0 \cos(2\phi), \quad \rho = r, R \]

- Inside of the material, there is a bulk equivalent current density

\[ \vec{J} = \vec{\nabla} \times \vec{M} \]

\[ J_z = \frac{3M_0}{\rho} \cos(2\phi) \]
Creating a “Green function”

- To solve for the fields due to these current arrays we first note that for a current sheet at $\rho=a$ with $\cos(2\phi)$ dependence, the solutions for the vector potential inside and outside of the sheet are

$$A_z = A_2 \cos(2\phi) \begin{cases} 
\rho^2, & \rho < a \\
\frac{a^4}{\rho^2}, & \rho > a 
\end{cases}$$

- The magnetic fields of this pure quadrupole are given by

$$\vec{B} = -2A_2 \begin{cases} 
\rho \sin(2\phi) \hat{\rho} + \rho \cos(2\phi) \hat{\phi}, & \rho < a \\
\frac{a^4}{\rho^3} \sin(2\phi) \hat{\rho} - \frac{a^4}{\rho^3} \cos(2\phi) \hat{\phi}, & \rho > a. 
\end{cases}$$
The discontinuity in the azimuthal field is created by the current sheet at $\rho=a$; 
\[ \Delta B_\phi(\phi) = \mu_0 K_z(\phi) = \mu_0 K_0 \cos(2\phi) \]
\[ = 4 A_2 a \cos(2\phi) \]

Thus, $A_2 = \frac{\mu_0 K_0}{4a}$ or in terms of the field gradient, 
\[ B' = 2A_2 = \frac{\mu_0 K_0}{2a} \propto \frac{K_0}{\rho} \]
for the single current sheet.

For a radially distributed current density $J_z = J_0(\rho) \cos(2\phi)$ (between $r_1$ and $r_2$), this result can be generalized to give 
\[ B' = \frac{\mu_0}{2} \int_{r_1}^{r_2} J_0(\rho) d\rho \]
Application to permanent magnet case

- The contributions due to the boundaries at $\rho = r_1, r_2$:

$$B' = \frac{\mu_0 M_0}{2} \left( \frac{1}{r_1} - \frac{1}{r_2} \right) = \frac{B_r}{2} \left( \frac{1}{r_1} - \frac{1}{r_2} \right)$$

where $B_r$ is the peak remanent magnetic field of the PM material.

- The contributions to the field gradient due to the bulk of the magnetic material are

$$B' = \frac{3\mu_0 M_0}{2} \int_{r_1}^{r_2} \frac{d\rho}{\rho^2} = \frac{3}{2} B_r \left( \frac{1}{r_1} - \frac{1}{r_2} \right)$$

- The total field gradient for the pure quadrupole is

$$B' = 2B_r \left( \frac{1}{r_1} - \frac{1}{r_2} \right)$$

For our case, a PMQ with 5 mm ID and $B_r=1.2$ T, we have $B' = 600$ T/m!
Segmented magnets

- The pure quadrupole is impossible to build, but may be well-approximated using uniformly magnetized pieces.
- Segmented slices (pizza-pie with bites taken out) can be analyzed using our methods.
- Fourier analysis gives equivalent Amperian current densities at desired multipole, and harmonics!

UCLA PMQ RADIA model

CESR collider final focus
PMQ cross-section
Fourier analysis of segmented PMQ

- $M$ segments used (e.g. 16 for our case)
- Interior region (straightforward, summation of $\delta$-functions)

\[
J_z = \frac{3}{\rho} \sum_{n=2,4,6...}^\infty a_n \cos(n\phi)
\]

\[
a_n = \frac{1}{2\pi} \sum_{m=1}^M \cos\left(\frac{2\pi}{M} \left[ m - \frac{1}{2} \right] \right) \cos\left(\frac{2\pi n}{M} \left[ m - \frac{1}{2} \right] \right)
\]

\[
a_n = \cos^n\left(\frac{\pi}{M}\right) \sin\left(\frac{n\pi}{M}\right) \frac{M}{n\pi}, \quad \text{where } n = 2 + jM, \ j = 0,1,2...
\]

- The inner/outer radii surfaces produce analogous results

This procedure approximates the sinusoidal magnetization with a series of $\delta$-functions.
PMQ strength and higher terms

- Fundamental strength is derated from perfect quadrupole by Fourier harmonics

\[ a_2 = \cos^2 \left( \frac{\pi}{M} \right) \sin \left( \frac{2\pi}{M} \right) \frac{M}{2\pi} \]

<table>
<thead>
<tr>
<th>M (segments)</th>
<th>( a_2 )</th>
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</thead>
<tbody>
<tr>
<td>8</td>
<td>0.77</td>
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<tr>
<td>12</td>
<td>0.89</td>
</tr>
<tr>
<td>16</td>
<td>0.94</td>
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- The next higher term is \( n=18! \) Relative strength is

\[ \frac{a_n}{a_2} = \cos^{n-2} \left( \frac{\pi}{M} \right) \left( \frac{1 - \left( \frac{r_1}{r_2} \right)^n}{1 - \left( \frac{r_1}{r_2} \right)^2} \right) \]

- Term is not large, and order is very high...
Field gradient model

- **Fixed RADIA solution** gives 550 T/m with $B_r=1.2$ T
- **Very good linearity**
- **Model prediction:** 600 T/m
- **Deviations due to:**
  - finite longitudinal (3D) effects
  - demagnetization
Measurements

- Hall probe scan: 550 T/m
- Consistency with beam focus tuning (TRACE3D): 550 T/m
- Pulsed wire: 475 T/m
  - Is pulse short enough for first integral? Needs to be a δ-function
  - Frequency dependence of wire

Calibration setup for PMQ gradient measurement