The energy loss and gain of a beam in the nonlinear, “blowout” regime of the plasma wakefield accelerator (PWFA), which features ultra-high accelerating fields, linear transverse focusing forces, and nonlinear plasma motion, has been asserted, through previous observations in simulations, to scale linearly with beam charge. In a new analysis that is the companion to this article, it has been shown that for an infinitesimally short beam, the energy loss is indeed predicted to scale linearly with beam charge for arbitrarily large beam charge. This scaling holds despite the onset of a relativistic, nonlinear response by the plasma, when the number of beam particles occupying a cubic plasma skin-depth exceeds that of plasma electrons within the same volume. This paper is intended to explore the deviations from linear energy loss using 2D particle-in-cell (PIC) simulations that arise in the case of finite length beams. The peak accelerating field in the plasma wave excited behind the finite-length beam is also examined, with the artifact of wave spiking adding to the apparent persistence of linear scaling of the peak field amplitude well into the nonlinear regime. At large enough normalized charge, the linear scaling of both decelerating and accelerating fields collapses, with serious consequences for plasma wave excitation efficiency. Using the results of parametric PIC studies, the implications of these results for observing the collapse of linear scaling in planned experiments are discussed.

1. Introduction

The scheme of using of electron beam-excited plasma waves, or plasma wakefields, to generate ultra-high accelerating gradients for future linear accelerators, is known as the plasma wakefield accelerator (PWFA). In the past decade, work on the PFWA has concentrated on extending the PWFA from the linear regime, where the plasma oscillations can be considered small perturbations about an equilibrium, to the highly nonlinear “blow-out” regime. In the blow-out regime, the plasma response to a beam that is much denser than the ambient plasma is violently nonlinear, as the plasma electrons are ejected from the path of the intense driving electron beam, resulting in an electron-rarefied region. This region contains only (nearly stationary) ions, and thus
possesses linear electrostatic focusing fields that allow high quality propagation of both the driving\textsuperscript{4,5,6} and accelerating beams. In addition, this region has superimposed upon it (TM) longitudinal electromagnetic fields, which, because the phase velocity of the wake wave is axisymmetric and nearly the speed of light, are independent of radial offset from the axis. Thus this wake may accelerate a trailing electron beam just as a traveling wave linac, with strong transverse focusing conveniently supplied by the plasma ions\textsuperscript{3}.

The beam dynamics in this scenario are termed linear, and are conceptually easy to understand using common tools. On the other hand, until recently, the plasma dynamics of the PWFA blow-out regime, with their extreme nonlinearity, have been only qualitatively understood, mainly through both fluid and particle-in-cell simulations\textsuperscript{7,8}. The companion work\textsuperscript{1} to this paper has made progress in moving toward an analytical understanding of the plasma response to very large beam charges.

Despite the lack of analytical models for the nonlinear plasma response, it has been noted in a variety of studies that the accelerating and decelerating fields associated with the blow-out regime obey a Cerenkov-like scaling\textsuperscript{9,10,11}. This scaling\textsuperscript{12} predicts that the fields which produce energy loss and gain in the system are proportional to the square of the characteristic maximum frequency in the system (the plasma frequency, $\omega_p^2$). The efficient excitation of an oscillatory system by a pulse occurs when the pulse is short compared with the oscillator period, requiring the rms pulse length $\sigma_z$ to obey the constraint $k_p \sigma_z \leq 2$. Thus linear scaling implies that the PWFA decelerating and (assumed proportional) acceleration fields should be related to the pulse length by $E_z \propto \sigma_z^{-2}$.

This prediction provides motivation for recent experiments that employ bunch compressors to reduce $\sigma_z$. In recent measurements with compressed beam at FNAL\textsuperscript{13}, the trailing portion of a 5 nC, 14 MeV, $\sigma_z=1.2$ mm, beam pulse was nearly stopped in 8 cm of $n_0 = 10^{14}$ cm$^{-3}$ plasma, a deceleration rate of over 150 MeV/m, obtained in the PWFA blow-out regime. Despite the nonlinearity of the plasma motion in this experiment, as well as other recent measurements at SLAC, the linear scaling of wakefields with respect to charge seems to have been well upheld\textsuperscript{14}.

On the other hand, in the future, beams compressed to ever shorter lengths will be employed in experimental scenarios. For example, in the context of SLAC E-164 experiment\textsuperscript{15} it has been proposed to use a beam that is compressed to as short as 12 $\mu$m in rms length, resulting in over 10 GeV/m acceleration gradients. This experimental test is quite important, as it is a milestone on the road to the so-called “after-burner” concept\textsuperscript{16}, in which one
may use a very high energy drive beam to double the energy of a trailing beam population, e.g. converting the SLC at SLAC from 50 GeV per beam to 100 GeV. The use of such short drive beams, and verification of scaling, could thus help realize such an ambitious goal.

The present work is concerned with two aspects of the problem of plasma wake-field acceleration and its scaling to ultra-high fields. The first is to show that the assertion of a linear relationship between the exciting beam charge and the amplitude of the excited wake-fields previously deduced from simulations does not hold when the charge is high enough. The second is to explore the implications of the approach to understanding the physics of the PWFA in the very nonlinear regime suggested by the accompanying work that analyzes an interesting and illuminating limit of this regime, that of an infinitesimally short (rms bunch length $\sigma_z \rightarrow 0$) driving beam. This exploration is accomplished through particle-in-cell simulations that serve to verify the analytical results, and extend them beyond the strict applicability of the analysis.

This companion analysis paper is concerned only with the energy loss of an infinitesimally short beam. We now recapitulate the main results of this exact (in the limit that $\sigma_z = 0$, and further assuming an ultra-relativistic beam velocity $v_b \equiv c$) analysis. It was found in this limiting case that the induced decelerating field at the beam is strictly proportional to the induced charge. This is despite the fact that the plasma response changes qualitatively as the normalized charge

$$\tilde{Q} = 4\pi k_p r_p N_0 = \frac{N_p k_p^3}{n_0} \tag{1}$$

is raised. Equation 1 indicates that $\tilde{Q}$ is the ratio of the beam charge to the plasma electron charge located within a volume of a cubic plasma skin-depth, $k_p^3$. Here, the plasma skin depth is defined as $k_p = c / \omega_p$, where the electron plasma frequency is $\omega_p = \sqrt{4\pi e^2 n_0 / m_e}$, and $n_0$ is the ambient plasma electron density.

In the analysis of our companion paper, all time intervals are normalized to $\omega_p^{-1}$ and spatial distances to $k_p^{-1}$. This analysis further normalizes all densities to $n_0$, velocities to $c$, momenta to $m_e c$, all current densities to $en_e c$, and all fields to $m_e c \omega_p / e$. All normalized variables are indicated by the tilde symbol, e.g. $\tilde{H}_q = eH_q / m_e c \omega_p$, and $\tilde{Q}$. In this regard, it should be noted that if $\tilde{Q} \ll 1$, the plasma response should be linear — all other normalized variables are small compared to unity. Nonlinear features appear in the response when $\tilde{Q}$ approaches or exceeds one.
Some of these nonlinear features can be anticipated easily; as noted before, the plasma electrons become rarefied from the beam channel. We can comment in this regard that it is assumed for the moment that the beam is both radially narrow \( k_p \sigma_r << 1 \) and short \( k_p \sigma_z < 2 \), and thus if \( Q \) exceeds unity then the ratio of the beam-to-plasma density, \( n_b / n_0 = Q (2\pi)^{3/2} k_p \sigma_z (k_p \sigma_r)^2 \), is greater than one. Under these assumptions, which are most often, but not always, obeyed in experimental conditions, the statement that \( Q \) is much greater than unity implies blow-out regime conditions, where the beam is much denser than the plasma. Under such conditions, the plasma electrons are then ejected from the beam channel. For present experiments in the blow-out regime\(^{13,14} \), the value of \( Q \) is in the range of 1.5-4.

It may also be anticipated that accompanying this large amplitude plasma density modulation implies a large velocity response in the plasma electrons themselves. In fact, it has been observed in simulations that the plasma electrons attain relativistic velocities as they are ejected from the beam channel\(^{18} \). What was not appreciated before the analysis presented in our companion paper is that a strong component of the imparted relativistic momentum impulse is predicted to be in the forward longitudinal (+z) direction. For infinitesimally short (longitudinal \( \delta \)-function) beams, the longitudinal momentum impulse is predicted to be increasingly dominant; the radial and longitudinal components are related by

\[
\Delta p_z = \frac{1}{2} \Delta p_r, \quad (2)
\]

where \( \tilde{p} = p / m_e c \) is the normalized momentum. In contrast, in the case of excitation of small amplitude plasma waves, the initial longitudinal acceleration of the plasma electrons due to the introduction of the beam pulse is always in the negative direction (-z).

The analysis presented in Ref. 1 quantitatively predicts the state of the plasma and fields directly behind the \( \delta \)-function beam (located at the zero of the variable \( \tau = \omega_p (t - z / v_p) \)), having uniform surface charge density up to a normalized radius \( \tilde{a} = k_p a \), The longitudinal electric field is obtained through a process that initially requires calculation of integral of the magnetic field

\[
H = \int \hat{H}_\theta d\tau \quad \text{during beam passage;}
\]

\[
H(\tilde{r}) = \frac{Q}{\pi \tilde{a}} \begin{cases} 
K_i(\tilde{a}) I_z(\tilde{r}) & (\tilde{r} < \tilde{a}) \\
K_i(\tilde{r}) I_z(\tilde{a}) & (\tilde{r} > \tilde{a}),
\end{cases} \quad (3)
\]
where $I_f$ and $K_f$ are modified Bessel functions.

Noting further that the radial momentum impulse is simply $Δp_r = H$, one has $Δp_z = \frac{1}{2} H^2$; the normalized velocity components associated with these momenta are $\tilde{v}_r = H \left(1 + \frac{1}{2} H^2 \right)$ and $\tilde{v}_z = \frac{1}{2} H^2 \left(1 + \frac{1}{2} H^2 \right)$. The equation of continuity for the plasma electrons further gives an enhancement of the plasma electron density immediately behind the driving beam, $\tilde{n} = \left(1 - \tilde{v}_z \right)^{-1} = 1 + \frac{1}{2} H^2$.

The combined effects of the crossed electric and magnetic fields causes a “snow-plowing” of the plasma electrons, resulting in enhanced density.

Thus the normalized transverse current density is given by $J_r = \tilde{n} v_r = H$, which is identical to the linear (non-relativistic) analysis. This linear-like scaling is caused by the fortuitous cancellation of two effects: the radial velocity is limited by relativistic effects, while the density grows via the snow-plow effect, exactly compensating for the reduced radial velocity response. Further, in the case of the $δ$-function beam, we have for the field directly behind the disk-like beam, in the limit $\tilde{a} \ll 1$,

$$\tilde{E}_z(\tilde{r}) \bigg|_{\tilde{r}=\tilde{a}} = \tilde{q} = \int_0^{\tilde{a}} \tilde{H}(\tilde{r'}) d\tilde{r'} = \frac{\tilde{q}}{\pi \tilde{a}^2} \left[1 - \tilde{a} K_1(\tilde{a}) J_0(\tilde{r}) \right], \quad \tilde{r} < \tilde{a}$$

(4)

The energy loss gradient associated with this field is given, for a $δ$-function beam, by one-half of the longitudinal force directly behind the beam$^{1,17}$, $\tilde{F} = \tilde{E}_z / 2$. It is predicted to be linear in charge regardless of the size of $\tilde{Q}$.

This scaling in energy loss for the limiting case of the $δ$-function beam has been suggested as an illuminating example that helps explain the persistence of linear-like scaling of both the deceleration of a drive beam in the blow-out regime of the PWFA, and in the subsequent available acceleration fields. In this paper, we quantitatively explore, with particle-in-cell (PIC) simulations using the 2D axi-symmetric codes OOPIC$^{8}$ and MAGIC$^{18}$, the dependence of these fields on $\tilde{Q}$.

In order to perform such an analysis, several definitions associated with the simulations must be introduced. The typical on-axis longitudinal field profile excited in the blow-out regime is shown in Fig. 1. Three measures of the field amplitudes are given in this figure: the well-behaved decelerating field inside of
the driving beam, the peak accelerating field, which is characterized by a narrow spike, and “useful” field, that which directly precedes the spike. The decelerating field is investigated in detail in this paper to make a strong connection to the analysis in Ref. 1. The second measure of acceleration given is termed “useful” because the acceleration associated with the spike is an extremely narrow region, with negligible stored energy, and therefore of very limiting use for efficiently accelerating a real beam. It will be seen that the scaling of plasma wake-field amplitudes, as measured in particular by the peak accelerating field spike, follows linear-like behavior well into the nonlinear regime. The mechanisms behind this anomalous scaling are explored, as are the ways in which they fail in the extremely nonlinear limit.

**Figure 1.** The longitudinal field profile given by PIC simulation for PWFA excitation in the blowout regime. The decelerating field, peak accelerating field, and a defined “useful field”, which avoids the narrow spike region, are indicated in the drawing.

---

2 One must take care that the PIC simulation parameters (mesh size and simulation particle number) are chosen to give well-behaved decelerating fields. Simulation noise problems become more serious when the beam charge is raised to $\hat{Q}$ larger than unity.
Before examining the scaling of the longitudinal fields in the nonlinear regime of the PWFA, we begin our discussion of simulation results by looking at the qualitative aspects of the plasma electron response. In this way, we can verify aspects of the predictions of the nonlinear theory given in Ref. 1 using a method that is independent of the fluid analysis employed therein.

![Diagram](image)

**Figure 2.** Configuration space from MAGIC cylindrically-symmetric PIC simulation with $Q=20$, and $k_p\sigma_z = 0.11$, and $k_p a = 0.2$, and beam center at $z=1.33$ cm. Color code indicates electron positions that have relativistic positive momenta $p_z/m_e c > 1$ in black, with all other plasma electrons colored red. The initially accelerated plasma electrons are just ahead of the blow-out region, where radial motion moves the electrons away from the beam channel.

2. **Aspects of the plasma response**

In order to explore the predictions concerning the nonlinear plasma response from the analytical $k_p\sigma_z \ll 1$ result, we have performed a series of simulations using the fully relativistic PIC codes, MAGIC and OOPIC. Two codes were used initially to check consistency; both codes gave essentially the same answers for all comparisons. Both codes were run with 15 GeV initial beam energy, to guarantee that the beam is ultra-relativistic, and to suppress transverse evolution of the beam distribution. The first investigation undertaken
using MAGIC concerned the validity of the physical model we have deduced from the analysis of Ref 1. In particular, as one never expects the snow-plow effect from linear theory, it is important this is observed. The two main characteristics of snow-plow are: 1) a forward velocity component, and 2) a plasma electron density increase, both occurring in the region directly behind the beam. These effects are noted in beams of moderate length \((k_p \sigma_z \sim 1)\), but in order to observe this effect most strongly, we next display the result of a short (to approach the \(\delta\)-function limit) beam simulation.

Both of the qualitative predictions are dramatically verified in Fig. 2, where we display a simulation with high charge, \(\tilde{Q} = 20\), that is very short, \(k_p \sigma_z = 0.11\), and is also narrow, \(k_p a = 0.2\). Note that the “shock front” shown in this case, which consists of electrons moving both forward and radially outward at relativistic speeds (the selected electrons must have \(\tilde{p}_z > 1\)) is not a representation of the initial disturbance, which is localized around the longitudinal position of the beam. The front is canted because many of the electrons in it that are located far from the axis originated quite close to the axis. In fact, one may expect that the leading edge of this front consists of particles that are ejected with relativistic transverse velocity, and thus the edge must have roughly a 45-degree angle with respect to the axis (recall that the electrons are launched by a source moving at nearly light speed). This angle may be verified from inspection of Fig. 2; note the difference in radial and longitudinal scales. The ultimate trajectory of these ejected electrons impacts the possible acceleration available in the wake-field behind the beam; we will return to this subject below.

In order to more quantitatively explore the predictions of the analysis given in Ref. 1, a series of additional simulations were undertaken with OOPIC, that had the following cylindrically-uniform beam shape: flat-top radial distribution of width \(k_p a = 0.2\) and flat-top longitudinal of length \(k_p l_z = 0.1\) (effectively much shorter than even in the example of Fig 2). This type of beam, which is as close as possible that we could obtain (given the constraints of numerical stability in the simulation) to the ideal \(\delta\)-function length used in the analysis of Ref. 1, was then scaled in charge \(\tilde{Q}\) upwards from 0.2, to 2, 20 and 200. To illustrate the relevant physical processes, we begin by the plotting plasma electron density as well the plasma longitudinal current density, for the \(\tilde{Q} = 200\) case, in Figs. 3. It can be seen from Fig. 3(a) that the plasma density is strongly snow-plowed in the vicinity of the driving beam, with a strong component of the longitudinal current density located there. Within this high forward-current region, the plasma density (Fig. 3(b)) is roughly 6 times the ambient density, indicating the severity of the snow-plow in this highly nonlinear case.
Figure 3. OOPIC simulation of $\hat{Q}=200$ case, drive beam having uniform distribution of width $\tilde{a} = 0.2$ and length $\tilde{l}_z = 0.1$. False-color (a) Plasma longitudinal current density and (b) plasma electron density (ambient level is at 15, indicated in blue).
Both the plasma electron density and the longitudinal currents show a strong localization around the rarefied region in Figs. 3. It can be seen that the plasma disturbance is surprisingly well-behaved even during such large amplitude motion. The nonlinear plasma motion coheres fairly well until the strong wave-breaking event located where the plasma electrons return to the axis. Note that portions of the disturbance propagate to large radial offset; these artifacts are associated with the electrons that are strongly ejected by the beam, and also with other regions having large longitudinal current density (both positive and negative).

In order examine the details of the immediate plasma response to the beam, we plot the current densities directly behind the driving beam that resulted from these simulations. This macroscopic quantity reflects both the density and velocity state of the plasma electrons, and in the analytical case $\tilde{J}_r$ also represented the magnetic field response.

It can be seen in Fig. 4 that the $\tilde{Q}=0.2$ case shows a small amount of longitudinal current density directly behind the beam, even though the beam-plasma system may be naively thought to be in the linear response regime. Inspection of our expressions for the velocity and current density indicate that this is not a completely linear system; the peak velocity induced at the beam edge is expected to be $0.15c$ even in this case. The nonlinearity arises even with a relatively small charge, because the peak beam density is 1.6 times $n_0$.

**Figure 4.** Current densities just behind driving beam with $\tilde{Q}=0.2$, uniform distribution of width $\tilde{a} = 0.2$ and length $\tilde{l}_z = 0.1$. Analytical prediction $\tilde{j}_z = \frac{1}{2} \tilde{j}_r^2$ from theory also shown.
Figure 5. Current densities just behind driving beam with $\tilde{Q}=2$, uniform distribution of width $\tilde{a}=0.2$ and length $\tilde{l}_z=0.1$. Analytical prediction $\tilde{j}_z=\frac{1}{4}\tilde{j}_r^2$ from theory also shown.

Figure 6. Current densities just behind driving beam with $\tilde{Q}=20$, uniform distribution of width $\tilde{a}=0.2$ and length $\tilde{l}_z=0.1$. Note that the currents have moved away from the beam distribution ($\tilde{r}>\tilde{a}=0.2$) significantly during the beam passage.
In the higher $\tilde{Q}$ cases, however, nonlinear effects are even more pronounced. In Figure 5, we plot, along with $\tilde{J}_z$ and $\tilde{J}_r$, the quantity which relates the two in the $\delta$-function beam limit, $\tilde{J}_z = \frac{1}{2} \tilde{J}_r^2$; the comparison is quite good. Note that in Fig. 4, the longitudinal current density $\tilde{J}_z$ was small enough that its amplitude was dominated by noise, and the comparison is not as good.

Finally, we show the current densities associated the very nonlinear $\tilde{Q}=20$ case in Fig. 6. Note that the current densities have moved away from the beam distribution ($\tilde{r} > \tilde{a} = 0.2$) significantly during the beam passage. This further indicates that the induced decelerating electric field should be smaller, as the radial currents associated with the induced $\tilde{E}_z$ are reduced by the diminishing of the coupling (by simple proximity arguments) to the driving electron beam charge. It is also striking to note that normalized current densities $\tilde{J}$ are well larger than unity — without the snow-plow enhancement of the plasma electron density, this is strictly forbidden. For the $\tilde{Q}=200$ case, $\tilde{J}_z$ is, as expected, even larger (over 11), and exceeds $\tilde{J}_r$ by a factor of two. Thus, as predicted by analytical results of Ref. 1 (see also our Eq. 2), the longitudinal current density eventually exceeds the radial current density when $\tilde{Q}$ becomes very large.

3. Deceleration and acceleration scaling studies: ultra-short beam

In order to evaluate the characteristics of the induced electric field driven by an ultra-short beam, we summarize the results of the parametric scan in $\tilde{Q}$ (from 0.02 to 200) in Fig. 7. This scan is an example of ideal scaling, in which the beam geometry and plasma density are held constant, while the charge is increased. In this figure, we plot the average on-axis deceleration experienced by the driving beam, and the associated prediction of linear theory, which in the short beam limit is given by $\tilde{F}_{dec} = (\tilde{Q}/2\pi\tilde{a}^2)\left[1 - \tilde{a}K_1(\tilde{a})\right]$. We choose the average deceleration as a relevant measure to accurately quantify the energy imparted to the plasma by the beam passage, and to connect with both the analytical $\delta$-function beam limit and also the case of longer, Gaussian beams discussed below.
In addition we also plot three measures of the accelerating field for each case: two from the simulation, the peak acceleration, and the "useful" acceleration; also, the peak acceleration from the predictions of linear theory, obtained (as was also done for the average deceleration) by performing a convolution integration\(^{17}\) over the drive beam, using the result of Eq. 4 as the Green function in the convolution, \(\tilde{F}_{\text{max}} = \left(\frac{Q}{\pi\tilde{\alpha}^2}\right)[1 - \tilde{\alpha}K_1(\tilde{\alpha})] \approx 2\tilde{F}_{\text{dec}}\). This procedure is of course not valid for the nonlinear response, but is only employed to extrapolate the predictions of linear theory.

Figure 7 shows some expected and some unexpected behavior. First, we note with satisfaction that until \(Q > 20\), the average deceleration observed in simulation is very close to that predicted by linear theory. This is a direct
verification of the extrapolation of the linear response of the decelerating field predicted by the analysis of the δ-function beam. Above $\hat{Q}=20$, the decelerating field is smaller than one expects from this limiting case, because (as already noted in the current response shown in Fig. 5) the plasma electrons may already be rarefying the near-beam region during the passage of the beam. This effect is not possible within the analytical model — the electrons do not notably change position within the time of the beam passage.

The peak, as well as the useful, accelerating field observed in the simulations is, according to Fig. 7, well below (nearly a factor of 2) that expected from linear theory, even for small $\hat{Q}$. This effect is due to nonlinear response having much to do with the very high density ($n_b=1.6 n_0$ even for $\hat{Q}=0.02$) in such a short beam. Aspects of the microscopic mechanisms for diminishing of the acceleration field are evident in Figs. 3, and are discussed in conjunction with the results shown in Fig. 8. This type of nonlinearity is not observed for small $\hat{Q}$ cases in the longer beam simulations discussed in the following sections — such cases agree quite well with linear theory. We now turn to our examination of these simulation results.

4. Ideal scaling with a Gaussian beam-plasma system

While the δ-function beam limit is relevant to verification of the theoretical analysis, it is not of highest practical interest in bunched beams, which generally have a Gaussian current distribution, $\rho_b(z) \propto \exp\left(-z^2 / 2\sigma_z^2\right)$. Further, it has often been argued that one should choose the plasma density such that $k_p\sigma_z \approx 1$ to optimize drive beam energy loss and accelerating beam energy gain in a PWFA. In order to explore the deviation in plasma response from the analytical ($k_p\sigma_z \rightarrow 0$) result, therefore, we have performed a series of OOPIC simulations. We again take the beam of radius $\tilde{a} = 0.2$ (again keeping the transverse beam profile uniform, to compare with the extrapolations of linear theory), and Gaussian current profile with $k_p\sigma_z = 1.1$. We have scanned the charge from $\hat{Q} = 0.02$ to 200, values indicating linear to very nonlinear cases.
Figure 8. The average normalized energy loss rate of $\tilde{F}_{\text{dec}} = e[|E_z|/m_e c \omega_p]$ of a Gaussian-current electron beam with $k_p \sigma_z = 1.1$, $\tilde{a} = 0.2$, as a function of $\tilde{Q}$, from linear theory (solid bold line) and self-consistent PIC simulation (circles); the peak accelerating field behind the beam, $\tilde{F}_{\text{max}} = e|E_{\text{max}}|/m_e c \omega_p$, from linear theory (solid fine line) and PIC simulation (squares); also the useful field for acceleration (diamonds).

In Fig. 8, the average on-axis decelerating field, as calculated from the simulation through $\left(2\pi \sigma_z\right)^{-1} \int eE_z(z) \exp\left(-z^2/2\sigma_z^2\right) dz$, again compares well with the linear theory prediction of $\left(\tilde{Q}/2\pi \tilde{a}^2\right) \left[1 - \tilde{a} K_1(\tilde{a})\right] \exp\left(-k_p^2 \tilde{a}^2\right)$, until $\tilde{Q}$ exceeds 2. It should be noted that $\tilde{Q}=2$ is just into the blow-out regime, as the beam is 2.3 times denser than the plasma. One in fact would expect that the linear prediction would begin to fail for $\tilde{Q}$ an order of magnitude smaller, but it does not, because of the snow-plow effect. The enhancement of the coupling due to snow-plowing of the plasma electrons is a reflection of a longer interaction time, since the plasma electron are traveling longitudinally at
relativistic speed, and thus stay in contact with the beam longer. Thus, even though the plasma electrons may move radially outward away from the beam, the beam-plasma coupling stays anomalously strong for moderately large \( \hat{Q} \) (less than 10). At very large \( \hat{Q} \), this coupling, as measured by the deceleration, diminishes notably — it is an order of magnitude smaller than predicted by extrapolated linear theory.

Examination of the acceleration amplitudes in the simulation shows several interesting features. The first is that if one relies on the peak as a measure of the acceleration, the spike that occurs at the back of the accelerating region misleadingly indicates linear-like response until \( \hat{Q} = 20 \). In fact, the spike in the peak field that we have discussed above is magnified in the more nonlinear cases, causing a field enhancement relative to linear theory for \( \hat{Q} \approx 1 \). This phenomenon partly explains why field saturation was not noted in previous simulation scans. Even with this masking effect, however, the accelerating peak still displays saturation when \( \hat{Q} > 1 \), with increasing severity for \( \hat{Q} > 100 \). Examination of the useful acceleration amplitude, however, indicates that the accelerating field response is diminished above \( \hat{Q} = 2 \), just as is found for the average deceleration. For \( \hat{Q} > 100 \), the efficiency of exciting the acceleration field is very low. In fact, the useful acceleration diminishes more rapidly than the drive beam deceleration for large \( \hat{Q} \). This effect is more noticeable in the ultra-short beam simulation shown in Fig. 7, where the useful acceleration is smaller than the deceleration for even moderate \( \hat{Q} \).

There is, in the linear regime, a fixed relationship between acceleration and deceleration that is dependent only on beam geometry, not charge. The question then arises: where is the energy lost by the drive beam going, if not into acceleration? The answer is hinted at by the enhancement of this effect in the ultra-short, dense beam case — much of the energy that is deposited into the plasma by the beam does not, for plasma electrons initially close to the axis, go into generation of simple wave motion, but into very large amplitude scattered motion. While such electrons may seem, if one concentrates only on smaller radial regions, to be ejected in a near-ballistic manner from the beam region, in fact, they eventually lose energy, but at a different, much longer time-scale than the main oscillation. These electrons, having a tight time-profile, and a non-trivial density, may be considered to make up subsidiary “beams”, which produce their own decelerating wakes. Since these wakes are far from the axis, and proceed with a long periodicity, such electrons do not contribute to building the initial accelerating wake-field, and their energy is effectively lost from the main component of wave system.
Figure 9. Surface plot of $E_z$ in OOPIC PWFA simulation with $\tilde{Q}=20$, (a) long-beam geometry as in Fig. 8, and (b) ultra-short beam geometry, as in Fig. 7.
The generation of large radial-amplitude particles can be noted in the momentum-coded configuration space plot of Fig. 2, and the density/current profiles of Fig. 3. Other aspects of this phenomenon are shown in Fig. 9, which displays the longitudinal electric field associated with \( \tilde{Q}=20 \) cases, for both the long-beam geometry (Fig. 9(a)) as in the simulations of Fig. 8 (similar to Fig. 2), and for the ultra-short beam (Fig. 9(b)) as in the simulations of Fig. 7. In the long-beam case, while there is negligible field disturbance electromagnetic energy density lost to large radial amplitudes emanating from the drive-beam region, there is a pronounced effect arising from the regions immediately following the field spikes that terminate the accelerating phases of the wake. These spikes are caused by electrons having very large momentum as they return to the axis, and this momentum is not removed during the dissolution of the spike. Thus many relativistic electrons are lost to subsequently large radial amplitudes. Associated with the wake of these electrons, which have a much shorter temporal length than the original drive beam, we again see significant levels of \( E_z \).

The ejection of plasma electrons from the drive beam region to large radial amplitude is more severe for larger \( \tilde{Q} \) cases. It is also a dominant effect in the ultra-short beam response, as seen in Fig. 9(b). A very large field response is observed propagating to large radial amplitudes in this case, one which clearly carries with it a significant fraction of the total electromagnetic energy deposited in the plasma (along with the mechanical energy associated with the motion). The severity of this effect for the ultra-short beam case illustrates the underlying mechanism behind the observation of accelerating field response that is lower than linear predictions, even for relatively small \( \tilde{Q} \).

5. Experimental scaling with a Gaussian beam-plasma system

The type of beam-plasma system scaling that has been explored in the simulations of Figs. 7 and 8 is what has been termed ideal. In performing these calculations, the plasma density and beam geometry were kept constant, while the charge was varied. This is of course exactly what one wants to do to conceptually understand the scaling of the system. It is not, however, what is strictly relevant to experiments. In accessing higher values of \( \tilde{Q} \) in experiment, the beam is compressed longitudinally\(^3\), and one typically then raises the plasma density to keep the condition \( k_p\sigma_z = 1 \). Thus \( \tilde{Q} = 4\pi k_p r_e N_0 \) is raised by

\(^3\)In many electron sources (e.g. rf photoinjectors with or without compressors) when one raises the charge, the bunch length also increases.
increasing $k_p$ with $N_b$ constant. In ideal scaling, the beam density increases as $k_p^3$, while $n_0$ increases only as $k_p^2$, thus leading to higher values of $n_b/n_0$.

In order to accomplish ideal scaling, it is implied that the beam’s matched beta-function $\beta_{eq}$ scales as $k_p^{-1}$, which is in fact the case\(^{3,4}\), as $\beta_{eq} = \sqrt{2\gamma k_p^{-1}}$. On the other hand, it is clear that the beam emittance $\varepsilon$ does not decrease during compression; it can be at best constant, and may indeed increase due to collective effects\(^{20,21}\) such as coherent synchrotron radiation. If one assumes that

\[ F_{\text{dec}} = \tilde{F}_{\text{dec}} = eE_z/m_e c \omega_p, \]

\[ F_{\text{max}} = \tilde{F}_{\text{max}} = eE_{\text{max}}/m_e c \omega_p, \]

\[ \tilde{\gamma} = 0.2 \sqrt{\tilde{Q}/2}, \]

\[ \tilde{Q} = \gamma k_p \]

\[ n_b/n_0 \]

\[ \beta_{eq} = \sqrt{2\gamma k_p^{-1}} \]
the beam emittance is held constant during compression, and that the beam is injected in a matched fashion to the ion focusing, then the normalized transverse beam size scales \( \tilde{\sigma}_r = k_p \sqrt{\beta_n} e \propto \sqrt{k_p} \).

The results of a parametric study using what may be therefore termed “experimental” scaling are shown in Fig. 10. In these simulations, the normalized bunch length is held constant, \( k_p \sigma_z = 1.1 \), while the normalized transverse (uniform density) beam size is given by \( \tilde{a} = 0.2 \sqrt{\tilde{Q}} / 2 \), as dictated by the constant emittance hypothesis. Note that because \( \tilde{a} \) is now a function of \( \tilde{Q} \), the extrapolated linear average deceleration and peak acceleration are no longer straight lines on the log-log plot. As the beam spot becomes larger with increasing \( \tilde{Q} \), the electromagnetic coupling of the beam to the plasma decreases, as could have been anticipated by inspection of Eq. 4.

One interesting point in this regard is that the beam no longer becomes relatively denser than the plasma as \( \tilde{Q} \) is increased — \( n_b / n_0 \propto \tilde{Q}^0 \), and the relative density is stationary. In the example of Fig. 10, this ratio is 2.3, at the lower end of the blowout regime, and strong blowout is never accessed. As \( \tilde{Q} \) is raised, however, the plasma response to the beam under experimental scaling also displays qualitatively different behavior — the plasma motion becomes increasingly longitudinal as \( \tilde{a} \) approaches and exceeds unity, with the plasma return currents running inside of the beam. When \( \tilde{a} \gg 1 \), one may expect that there is no relative reduction of beam coupling to the plasma with increasing charge, as beam does not eject the plasma electrons radially from its path. As such, the response in this limit does not resemble the blow-out regime, but a 1D nonlinear system that was discussed previous to the proposal of the blow-out regime.

These considerations shed light on the most striking aspect of the experimental scaling results of Fig. 10; the simulated deceleration and acceleration amplitudes are not notably different than those found from extrapolation of linear theory. Even the useful acceleration fields do not degrade from linear expectations by more than a factor of two. The electric field response in this study is well approximated by the linear prediction, since the beam is not too much denser than the plasma, and the plasma response is not radial motion-dominated as the normalized charge increases to nominally “nonlinear” levels.

This study illustrates one of the expected laboratory pitfalls of experimental scaling — the plasma does not become more underdense when the beam is compressed. Thus one may expect the longitudinal fields to increase with compression, and according to linear expectations. But one may not, as indicated
by simple arguments proposed by many previous authors, scale the expected field by $\sigma_z^{-2}$.

To illustrate this point, we take the case of the E164 experiment\textsuperscript{15}, now proposed for SLAC, as an extension to the E162 measurements\textsuperscript{14}. In the first stage of E164, the beam will be compressed from $\sim 700 \mu m$ to $\sim 100 \mu m$, with the plasma density raised to keep $k_p\sigma_z$ approximately constant. This jump in density represents a scaling in $\tilde{Q}$ from roughly 1.5 to near 10. An additional stage of E164 is more audacious, with the beam compressed to $\sim 12.5 \mu m$, and a concomitant $\tilde{Q}$ of $\sim 80$. The parameters chosen for the radial beam size in our experimental scaling scan also correspond well to those achieved at the SLAC FTFB ($k_p\sigma_z = 2\tilde{a} = 0.2$ at $\tilde{Q} = 2$), so this scan may be taken as a fairly accurate guideline for anticipating experimental scenarios. In fact, we have explicitly included the $\tilde{Q}=80$ point in order to compare the simulation to the approximate experimental conditions. Note that the scaling of $\tilde{Q}$ from 1.5 to 80 implies, using the linear extrapolation of field gradients (ideal scaling), that the acceleration should go from around 300 MeV/m, as is predicted by the results in Fig. 10, and was recently observed\textsuperscript{15}, to nearly 850 GV/m (!).

For the first stage of E164, experimental scaling the results of Fig. 10 gives a maximum acceleration gradient of 5 GV/m, which is slightly larger than claimed in Ref. 15. For the second stage of the E164 experiment, our scaling study predicts that the useful accelerating field is near 50 GV/m. This is an impressive number, but is over an order of magnitude shy of the prediction given by the $\sigma_z^{-2}$ extrapolation of linear theory. It also interesting to note in this context that if one could perform ideal scaling (by reducing the emittance along with bunch length), the nonlinearity of the plasma response, dominated by the radial ejection of the plasma electrons and concomitant loss of coupling, produces a useful acceleration which is nearly the same as in the experimental scaling case. Thus the nonlinear saturation of the plasma wave response with such a high value of $\tilde{Q}$ negates the advantage of using a narrower beam that is indicated by Eq. 4.

6. Conclusions

In conclusion, we have examined, through PIC simulation, aspects of the physics of the plasma electrons as they responds to a very high charge beam. We have concentrated on two areas of investigation: verification of the analytical results concerning the fluid response derived in Ref. 1, and the scaling of the beam-plasma interaction in both ideal and experimental scenarios.
The simulations that most directly dealt with the predictions of Ref. 1 were performed with an ultra-short \((k_p l_z=0.1)\) beam. The most important qualitative analytical predictions, those of a strong initial forward component of the plasma electrons, along with the associated increase in density (and current density) excited by a large \(\hat{Q}\) beam, were indeed verified by these simulations. In addition, the linear-like scaling of the deceleration fields driven by the beam was observed up to extremely large \(\hat{Q}\), with significant deviations entering in only when \(\hat{Q}\) exceeded 20. This is again in accordance with the most striking of the analytical results given in Ref. 1. This linear-like response in the fields stands in stark contrast to the nonlinear aspects (phenomena not associated with the linear theory) of the current density response that was observed even for \(\hat{Q}\) significantly smaller than unity. It is noted that the current density disturbance even for such a short beam leaves the immediate radial vicinity of the beam for \(\hat{Q}>>1\). This fast ejection of the plasma electrons results eventually in a loss of coupling between beam and plasma, and thus in lower wake-field amplitudes.

In order to connect the nonlinear physics observed in analysis and theory for the ultra-short beam case to more experimentally relevant scenarios, a series of studies were undertaken with beam-plasma systems that have \(k_p \sigma_z=1.1\). To explore the physics most characteristic of the PWFA blow-out regime, an ideal scaling study, where both \(k_p a\) and \(k_p \sigma_z\) were held constant while \(\hat{Q}\) was varied, was performed. In this case, the decelerating fields inside of the beam and the accelerating fields behind the beam displayed deviations from linear behavior. In particular, while the peak acceleration that is dominated by the spike at the tail of the accelerating portion of the wave scales nearly with linear expectations until \(\hat{Q}>>2\), the useful field for acceleration is strongly degraded from the expectations of linear theory above \(\hat{Q}=2\). The masking of this degradation by reliance on measurement of the acceleration amplitude through the spike is noted as a probable cause for missing the deviations from linear scaling in previous studies.

The ideal scaling study definitively showed that while the previously proposed scaling of wake-field amplitudes as linear with \(k_p^2\) (or \(\sigma_z^2\) for constant \(k_p a\) and \(k_p \sigma_z\)) is remarkably persistent until \(\hat{Q}\) well exceeds unity for finite length beams, when \(\hat{Q}\) is large enough, the coupling of the beam to the plasma becomes much less efficient, and field amplitudes do not grow as expected. Further, for very high \(\hat{Q}\), plasma electrons that are strongly ejected in the radial direction, and therefore do not contribute to creation of a useful initial accelerating portion of the wake wave, form an increasing component of the energy lost by the drive beam to the plasma. For these reasons, it may be unwise
remarkable.

electrons

long-beam

plasma

large

amount

would display significant degradation in amplitude due to nonlinear effects.

out

nitude

much

the

dances

beam

with normalized beam radius (or experimental predicted simulation taken

have

performed,

ambitious

case,

probably

implying

in

the

model.

be

one

creation

decrease.

that

is

to

the

beams,

for

the

limited

the

effective

beams

was

the

the

were

to

the

data

the

beams.

to

the

beams.

The

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.

beams.
microscopic physics of the plasma motion accompanying the formation of the accelerating phase of the wake may yield an explanation for such coherence.

Acknowledgments

This work supported by U.S. Dept. of Energy grant DE-FG03-92ER40693.

References

15. P. Muggli, *et al*., “Plasma wakefield experiments with 28.5 GeV electron and positron beams”, these proceedings.